

Local time on the exceptional set of dynamical percolation, and the Incipient Infinite Cluster

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Abstract

In dynamical critical site percolation on the triangular lattice or bond percolation on \mathbb{Z}^2 , we define and study a local time measure on the exceptional times at which the origin is in an infinite cluster. We show that at a typical time with respect to this measure, the percolation configuration has the law of Kesten's Incipient Infinite Cluster. In the most technical result of this paper, we show that, on the other hand, at the first exceptional time, the law of the configuration is different. We also study the collapse of the infinite cluster near typical exceptional times, and establish a relation between static and dynamic exponents, analogous to Kesten's near-critical relation.

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1 Introduction

Critical planar percolation is a central object of probability theory and statistical mechanics; see [Gri99, Wer09] for background. The best understood example is **Bernoulli**(1/2) site percolation on the triangular lattice \mathbb{T} , where conformal invariance and hence convergence of interfaces to SLE_6 is known [Smi01, Sch00, Smi06, CN07]. Nevertheless, many results are known for critical bond percolation on \mathbb{Z}^2 and other nice lattices, as well. In particular, almost everything in the present paper will apply equally to site percolation on \mathbb{T} and bond percolation on \mathbb{Z}^2 .

In **dynamical percolation**, a model introduced independently by [HägPS97] and Itai Benjamini, the status of each bit (site or bond) is continuously and independently resampled from the **Bernoulli**(p) measure, at times given by independent Poisson clocks of rate one. We will always consider site percolation on \mathbb{T} and bond percolation for \mathbb{Z}^2 , at the critical value $p = p_c = 1/2$. One of the principal reasons that dynamical percolation is interesting is that it provides a natural coupling of an uncountable number of copies of the underlying percolation process, and there may exist some exceptional instances of these copies that satisfy certain events that have zero probability in static percolation. The existence (or non-existence) of such **exceptional times** is called dynamical sensitivity (or stability) of the event, and the key event in question is of course the existence of an infinite cluster. See [Ste09] for a survey, but here is a brief summary of the subject. It was proved in [HägPS97] that for $p \neq p_c$ on any graph, both the existence and non-existence of infinite clusters are dynamically stable; then, dynamical stability of non-existence also holds at $p = p_c$ on \mathbb{Z}^d with $d \geq 19$ and on regular trees; and finally, there exist non-regular but spherically symmetric trees with no infinite clusters at p_c in static percolation, but with exceptional times in dynamical percolation. See [Kho08, PSS09] for more recent results on trees. The first example of dynamical sensitivity at p_c in a transitive graph was given by [SchSt10], proving it for the triangular lattice \mathbb{T} . This paper used discrete Fourier analysis, a tool that was introduced by [BKS99] for the closely related problem of noise sensitivity of percolation. This technique was further developed in [GPS10], proving that the set of exceptional times almost surely has Hausdorff dimension $31/36$, and showing dynamical sensitivity of critical percolation also for bond percolation on \mathbb{Z}^2 . Further studies of dynamical sensitivity and stability include [BS98, BrGS12, Ahi11] for percolation type processes, [BrS06, DCGP11] for Ising Glauber dynamics, and [BHPS03, Hof06, FNRS09] for some other processes.

The rare appearances of infinite structure at the exceptional times are reminiscent of the **Incipient Infinite Cluster**: a term used by physicists to refer to the large-scale connected structure present in critical percolation, and defined mathematically by Kesten as follows.

Definition 1.1. *The incipient infinite cluster, denoted by IIC , is the weak limit of the probability measures $\mathbb{P}_{p_c}(\cdot | 0 \leftrightarrow n)$ as $n \rightarrow \infty$, provided that the limit exists.*

Here, $\{0 \leftrightarrow n\}$ denotes the event that the open cluster of the origin reaches to distance n .

(We will formulate a precise definition shortly.) The existence of the IIC for numerous lattices in two dimensions was proved by Kesten [Kes86]. In high dimensions, properties of IIC and its scaling limits have been investigated in detail using the lace expansion [HarS00a, HarS00b]. In two dimensions, several other natural means of locating large structures at criticality — such as using the above definition with the condition $0 \leftrightarrow n$ replaced by the requirement that the open cluster of the origin have size at least n , or the weak limit as $n \rightarrow \infty$ of the largest cluster in $[-n, n]^2$ viewed from a uniformly chosen vertex in the cluster — have been shown to also be equal to IIC [Jar03]. These results support the view that, at least in dimension two, any natural means of selecting a limit of large scale critical structure is the IIC. One may ask then how the IIC may be found in dynamical percolation — and this question is central to the present paper.

1.1 The first exceptional time

There is one very natural means of selecting an exceptional time at which the cluster of the origin in dynamical percolation is infinite:

Definition 1.2. *Let \mathcal{E} denote the random set of times at which the cluster of the origin is infinite. We define the first exceptional time FET to be $\inf \mathcal{E} \cap (0, \infty)$. That $\text{FET} < \infty$ almost surely follows from the principal result of [SchSt10] for \mathbb{T} and from [GPS10] for \mathbb{Z}^2 . Note that FET is positive almost surely, since some positive time passes before there is a change in any bit (be it site or bond) in the boundary of the finite cluster of the origin in ω_0 . The law of ω_{FET} will be denoted by FETIC, the first exceptional time infinite cluster.*

Although it may be a natural candidate for the appearance of the incipient infinite cluster in dynamical percolation, FETIC is not the right choice:

Theorem 1.3. *The laws FETIC and IIC are not equal.*

Proving Theorem 1.3 is this paper’s most complex task. Roughly speaking, we show that the cluster of the origin under FETIC is somewhat thinner than under IIC. Indeed, as we will state more precisely in the next subsection, while the configuration at a “typical” exceptional time turns out to have the law of IIC, with many other exceptional times nearby, FET appears at the endpoint of a unit-order interval in which exceptional times are absent; in fact, finite approximations to FETIC may be constructed by size-biasing according to the length of the interval lacking connection from 0 to a high distance R leading up to a moment of such a connection. As such, FETIC assigns more mass to configurations which are liable to break apart easily under the perturbation provided by dynamical percolation. What makes the proof difficult is to detect this imbalance also in the limit $R \rightarrow \infty$. We will return to these vaguely sketched ideas when we prove Theorem 1.3 in Section 4.

It seems natural to suppose that the two measures differ to a greater degree:

Conjecture 1.4. *The measures FETIC and IIC are singular with respect to each other.*

The above intuitive explanation about how biasing by the length of the waiting time makes FETIC thinner than IIC might suggest that IIC stochastically dominates FETIC. However, IIC does not satisfy the FKG inequality (which we shortly review), and so it may be that such a general conclusion does not follow from the negative conditioning represented by longer waiting times.

Question 1.5. *Does IIC stochastically dominate FETIC?*

The invasion percolation cluster IPC is an infinite cluster associated to the critical point which is built by self-organized criticality. It was shown in [DSV09] that IIC and IPC are singular with respect to each other on \mathbb{Z}^2 . On the other hand, although IIC dominates IPC on regular trees [AGdHS08], this is not so on \mathbb{Z}^2 [Sap11].

It was pointed out to us by Alain-Sol Sznitman that, instead of considering the distribution at the first entry to a given subset of the state space in a Markov process, which is FETIC in our case, it is often more convenient to study the so-called equilibrium measure on the subset. For dynamical percolation on the ball B_R and the subset $\mathcal{A} := \{\zeta \in \{0,1\}^{B_R} : 0 \leftrightarrow R \text{ in } \zeta\}$, this measure is proportional at $\zeta \in \mathcal{A}$ to the probability that dynamical percolation started at ζ and stopped at an independent exponential time T leaves the set \mathcal{A} at the first update and does not return to it before T . The virtue of considering this measure could be that it has closer connections to the potential theory of the Markov process (Green's functions, Dirichlet forms, etc.; see [Szn11, Section 1.3]) than the first entry time, hence it might be easier to address the analogues of Theorem 1.3, Conjecture 1.4 and Question 1.5 for this measure.

1.2 The local time measure and the IIC

Our first effort to seek the IIC in dynamical percolation was hampered by biasing created by the procedure for selection. In light of this, it is natural to try again by considering the law of the configuration obtained by selecting an exceptional time at a “uniform” moment. However, this notion of uniformity requires more structure on the exceptional time set in order to make sense. For this reason, and because of its intrinsic interest, we construct a local time measure on the exceptional time set as a weak limit of normalized Lebesgue measure on the sets of connection time to large distances.

That is, we define a discrete local time μ_r for distance $r \in \mathbb{N}$ by setting

$$M_r(\omega_s) := \frac{\mathbb{1}\{0 \leftrightarrow r\}}{\mathbb{P}(0 \leftrightarrow r)}, \quad \text{and} \quad \mu_r[a, b] := \int_a^b M_r(\omega_s) ds. \quad (1.1)$$

We then wish to construct the following local time on \mathcal{E} as a limit of these measures:

$$\mu[a, b] := \lim_{r \rightarrow \infty} \mu_r[a, b]. \quad (1.2)$$

Theorem 1.6. *The limit (1.2) exists almost surely and in L^2 .*

This limit measure can be considered canonical also in the sense that, as we will see, another natural construction yields the same local time measure. So, we now have a measure from which we wish to sample uniformly to obtain a candidate for a law coinciding with IIC. However, μ is a σ -finite measure on \mathbb{R} so that further work is needed to make valid the notion of sampling a uniform point in the measure. The next two theorems give constructions of such a point and show that indeed the law of the configuration at the selected time is IIC.

Theorem 1.7 (Quenched sampling). *For almost every realization of the dynamical percolation process $\{\omega_s : s \in [0, \infty)\}$, and the corresponding local time measure μ , there exists some $T_0 < \infty$ such that for all $T > T_0$ we have $\mu[0, T] > 0$. Let χ_T be a random point from $[0, T]$ with law $\mu/\mu[0, T]$. Then, for almost all $\{\omega_s : s \in [0, \infty)\}$, the configuration $\omega(\chi_T)$ converges in law to IIC, as $T \rightarrow \infty$.*

Theorem 1.8 (Annealed sampling).

- (a) *For any fixed $T > 0$, let $\{\omega_s^* : s \in [0, T]\}$ be dynamical percolation reweighted (size-biased) by $\mu[0, T]$. Let χ_T^* be a random time from $[0, T]$ with law $\mu/\mu[0, T]$ for $\mu = \mu(\omega^*)$. Then, the configuration $\omega^*(\chi_T^*)$ has the distribution of the IIC.*
- (b) *Given a sample of $\mu = \mu(\omega)$ on \mathbb{R} , let Π_μ be the Poisson point process with intensity μ . One can make sense of conditioning $(\omega, \Pi_{\mu(\omega)})$ on $0 \in \Pi_{\mu(\omega)}$; this is called (ω^*, Π_μ^*) , the Palm version of (ω, Π_μ) . Then ω_0^* has the law of the IIC.*

A concrete means of realizing the Palm version of (ω, Π_μ) from dynamical percolation ω is Liggett's extra head construction, which we will describe in Section 3; see Figure 3.1.

Another application of the local time μ could be to run the dynamical percolation process ω according to $\mu(\omega)$. It should be possible to consider this time-changed dynamical percolation as a Markov process on configurations satisfying $0 \leftrightarrow \infty$, with stationary measure IIC; however, even the definition of the right state-space is unclear, especially if one wants IIC to be the unique stationary measure. We will not study these questions here.

1.3 Structure of the paper

We introduce our two discrete local time measures, μ_r and $\tilde{\mu}_r$, prove Theorem 1.6 for both, and prove the equality of the limits in Section 2. We then locate the IIC using the local time, proving Theorems 1.7 and 1.8 in Section 3. The more substantial Section 4 is devoted to telling apart FETIC and IIC, with a thinning procedure on bounded configurations being introduced and analysed in order to prove Theorem 1.3. The proof of Theorem 1.3 in fact exploits our identification of the IIC in dynamical percolation, because the proof considers a uniform right-hand endpoint of a period of connection $0 \leftrightarrow R$ and examines how long it takes for this connection to be reestablished as time advances; in finding an answer, we

will exploit the fact that the law of the configuration in B_R at this endpoint time is a close relative of critical percolation given $0 \leftrightarrow R$ (and thus also of IIC). Section 5 contains Theorem 5.1, a result addressing the question of how instances of the IIC embedded within dynamical percolation typically collapse as the time parameter is tuned at short distances to the moment at which the IIC appears.

As mentioned above, all our results apply equally to critical site percolation on the triangular lattice \mathbb{T} and critical bond percolation on \mathbb{Z}^2 , except for the existence and values of some critical exponents, of course, but we will formulate our results without using these exponents. For the sake of definiteness, we will work with critical site percolation on \mathbb{T} , or rather, with critical percolation on the faces of the dual hexagonal lattice.

1.4 Notation and percolation background

Let e_1 and e_2 denote the Euclidean unit vectors. The lattice in \mathbb{R}^2 with generators e_1 and $\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2$ induces a Voronoi tiling of the plane whose faces are hexagons. We refer to the set of these hexagons, with the adjacency relation given by two hexagons sharing a common edge, as the **hexagonal lattice** \mathcal{H} . The hexagon centred at the origin will be denoted by 0 . Note that the set of hexagons intersecting the x -axis forms a bi-infinite simple path. Define $d : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{N}$ to be graphical distance and set $B_R = \{h \in \mathcal{H} : d(0, h) \leq R\}$ for $R \in \mathbb{N}$. For $R_1, R_2 \in \mathbb{N}$ such that $R_1 < R_2$, write $A_{R_1, R_2} = B_{R_2} \setminus B_{R_1}$ for the annulus with inner and outer radii R_1 and R_2 . The (outer) boundary of a set $A \subset \mathcal{H}$ is $\partial A := \{h \in \mathcal{H} \setminus A : d(h, A) = 1\}$.

In critical percolation on \mathcal{H} , each $h \in \mathcal{H}$ is independently open or closed with probability one-half. The set $\{0, 1\}^{\mathcal{H}}$ of percolation configurations is equipped with the usual product topology, and the events are the subsets $\mathcal{A} \subseteq \{0, 1\}^{\mathcal{H}}$ that are measurable with respect to the corresponding Borel sigma-algebra. For $a, b \in \mathcal{H}$, we write $a \leftrightarrow b$ for the event that an open path of hexagons connects a and b . For $A, B \subseteq \mathcal{H}$, we write $A \leftrightarrow B$ if there exist $a \in A$ and $b \in B$ such that $a \leftrightarrow b$. For $R_1, R_2 \in \mathbb{N}$ such that $1 \leq R_1 < R_2$, we write $R_1 \leftrightarrow R_2$ to indicate that $\partial B_{R_1} \leftrightarrow \partial B_{R_2}^c$. For $R \in \mathbb{N}$, we also write $0 \leftrightarrow R$ for $0 \leftrightarrow \partial B_R^c$.

The open cluster of 0 , $\{h \in \mathcal{H} : 0 \leftrightarrow h\}$, will be denoted by \mathcal{C}_0 .

We will use the notation $\alpha_1(R_1, R_2) := \mathbb{P}(R_1 \leftrightarrow R_2)$ and $\alpha_1(R) := \alpha_1(1, R)$, this being the **one-arm** probability. Furthermore, $\alpha_4(R_1, R_2)$ denotes the **alternating four-arm** probability: the probability that there are two open and two closed paths connecting ∂B_{R_1} and $\partial B_{R_2}^c$, in an alternating order: open-closed-open-closed. Again, $\alpha_4(R) := \alpha_4(1, R)$.

Given a percolation configuration $\omega \in \{0, 1\}^{\mathcal{H}}$ and an event $\mathcal{A} \subseteq \{0, 1\}^{\mathcal{H}}$, we call a hexagon h **pivotal** for \mathcal{A} in ω if changing the status of h changes the outcome of the event. The set of pivotal hexagons will be denoted by $\text{Piv}_{\mathcal{A}}(\omega)$. For instance, note that h is pivotal for the left-right crossing event in a rectangular region of \mathcal{H} if and only if there are four alternating arms connecting h to the corresponding sides of the rectangle.

Let us now recall some standard tools in percolation theory [Wer09].

The Harris-FKG inequality. The set $\{0, 1\}^{\mathcal{H}}$ of percolation configurations on the hexagonal lattice has a natural partial order \leq . A percolation event $\mathcal{A} \subseteq \{0, 1\}^{\mathcal{H}}$ is called increasing if $\omega \in \mathcal{A}$ and $\omega \leq \omega'$ implies that $\omega' \in \mathcal{A}$. The inequality of Harris and Fortuin-Kesteleyn-Ginibre states that if \mathcal{A} and \mathcal{B} are increasing events, then $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B})$.

RSW estimates. For any $L > 0$, there exists a constant $c_L > 0$ such that the probability of an open path in critical percolation between the left and right sides of the region $\mathcal{H} \cap [0, Ln] \times [0, n]$ is at least c_L , independently of n .

Quasi-multiplicativity of arm probabilities. For $\ell \in \{1, 4\}$, there exists a constant $0 < c_\ell$ such that, for any radii $R_1 < R_2 < R_3$, we have

$$c_\ell < \frac{\alpha_\ell(R_1, R_3)}{\alpha_\ell(R_1, R_2) \alpha_\ell(R_2, R_3)} \leq 1. \quad (1.3)$$

The right-hand inequality is trivial; for $\ell = 1$, the left-hand one is a simple consequence of FKG and RSW; for $\ell = 4$, more work is needed, done in [Kes87a]; see also [Nol08, SchSt10]. Similarly to quasi-multiplicativity, one can show that we lose only a constant factor in probability if we require our four alternating arms to have their endpoints on nice prescribed arcs of the boundary. Together with some simple results on arm probabilities in the half-plane, this implies the following bounds on the number of pivotals: if $\mathcal{A}(R)$ is the left-right crossing event in the square $[0, R]^2$, then $|\text{Piv}_{\mathcal{A}(R)}| \asymp \alpha_4(R) R^2$, and if $\mathcal{A}(R_1, R_2)$ is the annulus crossing event $R_1 \leftrightarrow R_2$ with $R_1 < R_2/2$, then $|\text{Piv}_{\mathcal{A}(R_1, R_2)}| \asymp \alpha_4(R_2) R_2^2$, with constant factors independent of R_1 .

For critical percolation on \mathcal{H} , we also know the existence and values of critical exponents: $\alpha_1(R_1, R_2) = (R_1/R_2)^{5/48+o(1)}$ by [LSW02], and $\alpha_4(R_1, R_2) = (R_1/R_2)^{5/4+o(1)}$ by [SmW01], as $R_2/R_1 \rightarrow \infty$. In particular, $|\text{Piv}_{\mathcal{A}(R)}| = R^{3/4+o(1)}$ as $R \rightarrow \infty$. On \mathbb{Z}^2 , we have the bounds

$$C^{-1} (r/R)^{2-\eta} \leq \alpha_4(r, R) \leq C (r/R)^{1+\eta} \quad (1.4)$$

for some fixed constants $C > 0$, $\eta \in (0, 1)$ and every $1 \leq r \leq R$. See [GPS10, Eq. (2.6)] and the references there. Consequently, with some different value of the constant C ,

$$C^{-1} R^\eta \leq |\text{Piv}_{\mathcal{A}(R)}| \leq C R^{1-\eta}. \quad (1.5)$$

The near-critical window. One can consider monotone versions of dynamical percolation, in which dynamical updates lead always either to the closure or to the opening of hexagons. These give couplings between dynamical and off-critical percolation (and also a coupling of percolation measures at different densities), and therefore information on off-critical percolation can yield bounds on dynamical percolation questions. We will use these relations (which turn out to be sharp) several times.

Kesten found the near-critical window of percolation precisely [Kes87a] (see [Nol08, Wer09] for more modern accounts): for a system of linear size R , the window is given by

the reciprocal of the expected number of pivotals for the left-right crossing event $\mathcal{A}(R)$ at criticality. More precisely, for the annulus crossing event $\mathcal{A}(R, 2R)$, as $R \rightarrow \infty$, we have

$$\frac{\mathbb{P}_{p_c \pm \epsilon}(\mathcal{A}(R, 2R))}{\mathbb{P}_{p_c}(\mathcal{A}(R, 2R))} \rightarrow 1 \quad \text{if } \epsilon \ll \frac{1}{|\text{Piv}_{\mathcal{A}(R)}|}, \quad (1.6)$$

while

$$\delta < \mathbb{P}_{p_c \pm \epsilon}(\mathcal{A}(R, 2R)) < 1 - \delta \quad \text{if } \epsilon \asymp \frac{1}{|\text{Piv}_{\mathcal{A}(R)}|}, \quad (1.7)$$

with $\delta \in (0, 1)$ depending only on the constant factors giving the size of ϵ , and finally,

$$\mathbb{P}_{p_c + \epsilon}(\mathcal{A}(R, 2R)) \rightarrow \begin{cases} 1 & \text{if } \epsilon \gg \frac{1}{|\text{Piv}_{\mathcal{A}(R)}|}, \\ 0 & \text{if } -\epsilon \gg \frac{1}{|\text{Piv}_{\mathcal{A}(R)}|}. \end{cases} \quad (1.8)$$

Kesten also proved the stability of one- and alternating four-arm probabilities inside the window:

$$\begin{aligned} \frac{\mathbb{P}_{p_c \pm \epsilon}(\mathcal{A}_\ell(1, R))}{\mathbb{P}_{p_c}(\mathcal{A}_\ell(1, R))} &\rightarrow 1 \quad \text{if } \epsilon \ll \frac{1}{|\text{Piv}_{\mathcal{A}(R)}|}, \\ &\asymp 1 \quad \text{if } \epsilon \asymp \frac{1}{|\text{Piv}_{\mathcal{A}(R)}|}, \end{aligned} \quad (1.9)$$

for $\ell \in \{1, 4\}$. The $\epsilon \ll 1/|\text{Piv}_{\mathcal{A}(R)}|$ case of (1.9) and (1.6) are not stated explicitly in [Kes87a], but they clearly follow from his proof using differential inequalities.

Using the stability of the 1-arm and 4-arm probabilities in the near-critical window, he also found the off-critical exponent, a relation usually called Kesten's scaling relation [Kes87a, Corollary 1]:

$$\mathbb{P}_{p_c + \epsilon}(0 \longleftrightarrow \infty) \asymp \alpha_1(\rho(1/\epsilon)), \quad (1.10)$$

where $\rho(r) := \inf\{s \in \mathbb{N}_+ : s^2 \alpha_4(s) \geq r\}$ for $r \geq 1$, the inverse function of $R \mapsto |\text{Piv}_{\mathcal{A}(R)}|$. We have $\rho(r) = r^{4/3+o(1)}$ on \mathcal{H} , and $C^{-1}r^\eta \leq \rho(r) \leq Cr^{1/\eta}$ for some $0 < \eta, C < \infty$ on \mathbb{Z}^2 , by (1.5). Note here that Kesten formulated his result in terms of critical exponents, which would not be enough for us later because of the unspecified $o(1)$ terms in the exponent, but the proof clearly gives the stronger result we stated; see [Wer09, Chapter 6].

Dynamical percolation and a dynamical FKG inequality. As mentioned above, we will consider dynamical critical percolation with updates from the stationary distribution (resampling the bits) at times given by Poisson clocks of rate one, with time indexed by \mathbb{R} , and, just for the sake of definiteness, with càdlàg trajectories.

We will need the following extension of the FKG inequality to increasing events of dynamical percolation, an immediate consequence of [Lig05, Corollary II.2.21]. A weaker form (with a very different proof) was given in [HamMP12, Lemma 4.2].

Lemma 1.9 (Dynamical FKG inequality). *Let $\omega, \omega' : \mathcal{H} \times \mathbb{R} \rightarrow \{0, 1\}$ denote two realizations of dynamical percolation on the hexagonal lattice \mathcal{H} . We say that $\omega \leq \omega'$ if*

$\omega_t(x) \leq \omega'_t(x)$ for all $(x, t) \in \mathcal{H} \times \mathbb{R}$. Let $\mathcal{A}, \mathcal{B} \subseteq \{0, 1\}^{\mathcal{H} \times \mathbb{R}}$ be two increasing events (i.e., if $\omega \in \mathcal{A}$ and $\omega \leq \omega'$, then $\omega' \in \mathcal{A}$). Then $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B})$.

The same holds if the dynamics is not stationary, but started at time 0 from an arbitrary distribution on $\{0, 1\}^{\mathcal{H}}$ that satisfies the static FKG inequality.

Proof. Corollary II.2.21 of [Lig05] states this for increasing events that depend on the configuration at finitely many time instances $t_1 < \dots < t_n$, proved using induction on n and the infinitesimal generator of the process. Since all measurable dynamical events can be approximated by events depending on finitely many time instances, our statement follows. \square

1.5 The Fourier spectrum of critical percolation

A key tool for the analysis of dynamical percolation is discrete Fourier analysis. Here we provide the definition of the Fourier spectrum of a percolation event, explain the basic relation between the spectrum and decorrelation for the event under dynamical percolation, and collect the results from the literature that we will use. A far more thorough overview of this theory is provided by the survey article [GS12].

Let \mathcal{A} denote a percolation event in B_R , so that \mathcal{A} is a subset of percolation configurations in B_R . Define the usual inner product on the L^2 -space on percolation configurations on B_R by $\langle f, g \rangle = \mathbb{E}(fg) = 2^{-|B_R|} \sum_{\omega \in \{-1, 1\}^{B_R}} f(\omega)g(\omega)$, and note that the collection $\{\chi_S := \prod_{i \in S} \omega(i) : S \subseteq B_R\}$ is an orthonormal basis for this L^2 -space. As such, the $\{-1, 1\}$ -indicator function $f_{\mathcal{A}}$ of \mathcal{A} has a Fourier decomposition $f_{\mathcal{A}} = \sum_{S \subseteq B_R} \widehat{f}_{\mathcal{A}}(S) \chi_S$. Parseval's identity $\sum_{S \subseteq B_R} \widehat{f}_{\mathcal{A}}^2(S) = 1$ allows us to define a random variable $\text{Spec}_{\mathcal{A}}$, the spectral sample of \mathcal{A} , on subsets of B_R according to $\mathbb{P}(\text{Spec}_{\mathcal{A}} = C) = \widehat{f}_{\mathcal{A}}^2(C)$ for $C \subseteq B_R$.

Recall that the dynamical percolation process $\{\omega_t\}_{t \in \mathbb{R}}$ is defined using i.i.d. rate one Poissonian updates for each bit. Now, the basic relation between the spectral sample and decorrelation under this dynamics is that, for percolation events \mathcal{A} and \mathcal{B} in B_R ,

$$\mathbb{E}(\omega_0 \in \mathcal{A}, \omega_t \in \mathcal{B}) = \sum_{S \subseteq B_R} \widehat{f}_{\mathcal{A}}(S) \widehat{f}_{\mathcal{B}}(S) e^{-t|S|}. \quad (1.11)$$

This shows that if most of the measure for at least one of the spectral samples $\text{Spec}_{\mathcal{A}}$, $\text{Spec}_{\mathcal{B}}$ is supported on large sets S , then fast decorrelation occurs.

The spectral sample $\text{Spec}_{\mathcal{A}}$ is a random subset of B_R with some similarities to, and marked differences from, the random set $\text{Piv}_{\mathcal{A}}$ of hexagons in B_R that are pivotal for the occurrence of \mathcal{A} under critical percolation. As first observed by Gil Kalai, the two random variables share their first and second moments (see [GPS10, Section 2.3]),

$$\mathbb{E}|\text{Piv}_{\mathcal{A}}| = \mathbb{E}|\text{Spec}_{\mathcal{A}}|, \quad \mathbb{E}|\text{Piv}_{\mathcal{A}}|^2 = \mathbb{E}|\text{Spec}_{\mathcal{A}}|^2, \quad (1.12)$$

but not the higher ones, and their large deviations usually differ (see [GPS10, Remark 4.6]).

Of particular import to us is the case where \mathcal{A} is a crossing event from one boundary arc to another in some planar domain. Let us first consider $\mathcal{A}(R, 2R) = \{R \leftrightarrow 2R\}$. A standard second moment argument yields the conclusion that there exists $C > 0$ such that, for all R , $\mathbb{E}|\text{Piv}_{\mathcal{A}(R, 2R)}|^2 \leq C \mathbb{E}|\text{Piv}_{\mathcal{A}(R, 2R)}|^2$. In light of (1.12) and the second moment method, we see that for some small $c > 0$,

$$\mathbb{P}(|\text{Spec}_{\mathcal{A}(R, 2R)}| \geq c \mathbb{E}|\text{Spec}_{\mathcal{A}(R, 2R)}|) \geq c.$$

Thus, (1.12) shows that, for each $s > 0$, there exists $c(s) < 1$ (with the supremum of $c(s)$ strictly less than one over any interval of the form (ϵ, ∞)) such that, for all $R \in \mathbb{N}$,

$$\mathbb{P}(\omega_0 \in \mathcal{A}(R, 2R), \omega_t \in \mathcal{A}(R, 2R)) \leq c(s) \quad \text{where } t = s \mathbb{E}|\text{Piv}_{\mathcal{A}(R, 2R)}|; \quad (1.13)$$

thus, the characteristic time-scale for at least partial decorrelation of the crossing event is determined by the mean number of pivotals. We will also need the much stronger assertion, proved in [GPS10], that, as $s \rightarrow \infty$,

$$\mathbb{P}(\omega_0 \in \mathcal{A}(R, 2R), \omega_t \in \mathcal{A}(R, 2R)) \rightarrow 0 \quad \text{where } t = s \mathbb{E}|\text{Piv}_{\mathcal{A}(R, 2R)}|, \quad (1.14)$$

uniformly in $R \in \mathbb{N}$; on \mathcal{H} we have the sharp upper bound $s^{-2/3+o(1)}$. That is, the crossing event in fact decorrelates fully at large multiples of the scale determined by the mean pivotal number. The bound (1.14) arises from a detailed examination of the lower-tail of the size $|\text{Spec}_{\mathcal{A}(R, 2R)}|$ of the spectral sample.

Similar sharp results are proved in [GPS10] for the decorrelation of the crossing events $\mathcal{A}(0, R) = \{0 \leftrightarrow R\}$, which are the key for the applications to exceptional times. Namely, [GPS10, Equation (9.2)] says that, for all $s, t \geq 0$,

$$\frac{\mathbb{P}(\mathbb{1}\{0 \xleftrightarrow{\omega_s} R\} \mathbb{1}\{0 \xleftrightarrow{\omega_t} R\})}{\mathbb{P}(0 \longleftrightarrow R)^2} \leq O(1) \frac{1}{\alpha_1(\rho(1/|t-s|))} \quad (1.15)$$

$$\leq O(1) |s-t|^{-1+\delta+o(1)} \quad (1.16)$$

for some $\delta > 0$, uniformly in $R \in \mathbb{N}_+$, the $o(1)$ term being understood as $|s-t| \rightarrow 0$. On \mathcal{H} , also the sharp result $\delta = 31/36$ is known. For exceptional times, the importance of these decorrelation bounds lies in the fact that the exponent δ of (1.16) is a lower bound on the Hausdorff dimension of the set \mathcal{E} , using the so-called Mass Distribution Principle.

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2 Constructing the local time

In this section, we present the proof of Theorem 1.6. We begin by examining some basic properties of the discrete local time measures $\mu_r[a, b]$, defined in (1.1).

Note that $M_r(\omega)$ is a martingale with respect to the filtration \mathcal{F}_r of the percolation space generated by the variables $\mathbb{1}\{0 \leftrightarrow r\}$: for $r' > r$,

$$\mathbb{E} \left(\frac{\mathbb{1}\{0 \leftrightarrow r'\}}{\mathbb{P}(0 \leftrightarrow r')} \middle| \mathcal{F}_r \right) = \frac{\mathbb{P}(0 \leftrightarrow r' | 0 \leftrightarrow r)}{\mathbb{P}(0 \leftrightarrow r')} \mathbb{1}\{0 \leftrightarrow r\} = \frac{\mathbb{1}\{0 \leftrightarrow r\}}{\mathbb{P}(0 \leftrightarrow r)}.$$

Consequently, $\mu_r[a, b]$ is a martingale with respect to $\mathcal{F}_r[a, b]$, with $\mathbb{E}\mu_r[a, b] = b - a$, and thus we have a hope to apply some martingale convergence theorem to show that the limit over r exists in some sense and gives a non-degenerate random measure.

To paraphrase the outline of our approach to constructing the local time μ , we have selected a straightforward measure of the degree to which a configuration in the finite ball B_r aids in forming a connection from 0 to ∞ . This measure is a normalization of the indicator function on a connection $0 \longleftrightarrow r$. After integration in time, we obtain a discrete local time. This means of quantifying the benefit provided by a given finite configuration in enabling $0 \longleftrightarrow \infty$ is by no means the only natural choice and indeed we now introduce another, for which the accompanying sequence of discrete local times is a useful tool for establishing the existence of the limiting local time on \mathcal{E} .

Instead of postulating that a large radius of \mathcal{C}_0 is the most important indication of being close to $0 \leftrightarrow \infty$, we can look at how much a finite piece of the configuration actually helps in realizing a connection to infinity. For any finite set H of hexagons, we let ω^H denote the restriction of ω to H , and define the random variable

$$\tilde{M}_H(\omega) := \lim_{R \rightarrow \infty} \frac{\mathbb{P}(0 \leftrightarrow R | \omega^H)}{\mathbb{P}(0 \leftrightarrow R)}. \quad (2.1)$$

Of course, it is not at all obvious that the limit over R exists. However,

$$\frac{\mathbb{P}(0 \leftrightarrow R | \omega^H)}{\mathbb{P}(0 \leftrightarrow R)} = \frac{\mathbb{P}(0 \leftrightarrow R, \omega^H)}{\mathbb{P}(0 \leftrightarrow R)\mathbb{P}(\omega^H)} = \frac{\mathbb{P}(\omega^H | 0 \leftrightarrow R)}{\mathbb{P}(\omega^H)}, \quad (2.2)$$

whose right-hand side indeed has a weak limit in high R — this is nothing other than the IIC, whose construction was carried out in dimension two by Kesten. Thus, the limit in (2.1) indeed exists, so that we may define

$$\tilde{M}_r(\omega_s) := \tilde{M}_{B_r}(\omega_s), \quad \tilde{\mu}_r[a, b] := \int_a^b \tilde{M}_r(\omega_s) ds. \quad (2.3)$$

We then seek to show the existence of the limit $\tilde{\mu}[a, b] := \lim_{r \rightarrow \infty} \tilde{\mu}_r[a, b]$. In this case, (2.1) shows that $\tilde{M}_r(\omega)$ is a martingale with respect to the full filtration $\tilde{\mathcal{F}}_r$ generated by $\omega(B_r)$, and $\tilde{\mu}_r[a, b]$ is a martingale with respect to $\tilde{\mathcal{F}}_r[a, b]$, again with $\mathbb{E}\tilde{\mu}_r[a, b] = b - a$.

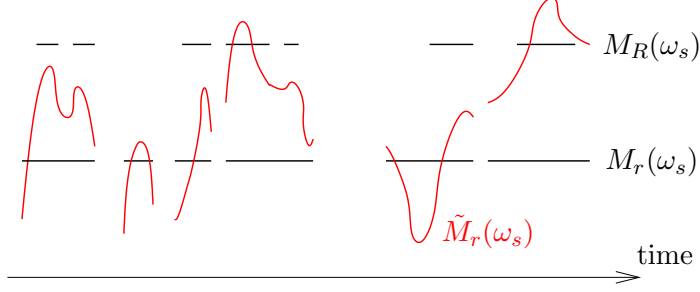


Figure 2.1: Schematic pictures of the approximate local time densities for μ and $\tilde{\mu}$.

Theorem 2.1. *Both limits, (1.1) and (2.3), exist almost surely and in L^2 , and the two local time measures almost surely coincide: $\mu[a, b] = \tilde{\mu}[a, b]$ for all intervals $[a, b]$ simultaneously.*

Proof. The main task is to prove the statements for any fixed interval $[a, b]$. This implies the claims for all $(a, b) \in \mathbb{Q}^2$ simultaneously, and then, since rational intervals generate the Borel sigma-algebra, we have the simultaneous statement for all $(a, b) \in \mathbb{R}^2$.

First recall the quasi-multiplicativity relation (1.3), which implies, for $R > r > 0$,

$$\frac{\mathbb{P}(0 \leftrightarrow R | \omega^{B_r})}{\mathbb{P}(0 \leftrightarrow R)} \asymp \frac{\mathbb{P}(0 \leftrightarrow R | \omega^{B_r})}{\mathbb{P}(0 \leftrightarrow r)\mathbb{P}(r \leftrightarrow R)} \leq \frac{\mathbb{P}(r \leftrightarrow R | \omega^{B_r})\mathbb{1}\{0 \leftrightarrow r\}}{\mathbb{P}(0 \leftrightarrow r)\mathbb{P}(r \leftrightarrow R)} = \frac{\mathbb{1}\{0 \leftrightarrow r\}}{\mathbb{P}(0 \leftrightarrow r)}.$$

Therefore, with an absolute constant $C_1 < \infty$,

$$\tilde{M}_r(\omega) \leq C_1 M_r(\omega) \quad \text{and} \quad \tilde{\mu}_r[a, b] \leq C_1 \mu_r[a, b]. \quad (2.4)$$

Secondly, recall from (1.16) the bound $O(1)|s - t|^{-1+\delta+o(1)}$, with $\delta > 0$. Integrating over s and t , this gives the second moment estimate

$$\mathbb{E}(\mu_r[a, b]^2) \leq \begin{cases} |b - a|^{1+\delta+o(1)} & \text{as } |b - a| \rightarrow 0, \\ C_2 |b - a| & \text{for all } a < b, \end{cases} \quad (2.5)$$

uniformly in r , with an absolute constant $C_2 < \infty$. Therefore, by (2.4), the L^2 martingale convergence theorem implies the existence of the limits

$$\mu_r[a, b] \xrightarrow[L^2]{\text{a.s.}} \mu[a, b] \quad \text{and} \quad \tilde{\mu}_r[a, b] \xrightarrow[L^2]{\text{a.s.}} \tilde{\mu}[a, b], \quad (2.6)$$

as we claimed.

Now, if we fix $r > 0$, and take $r' \rightarrow \infty$, then

$$\mathbb{E}(M_{r'} | \tilde{\mathcal{F}}_r) = \frac{\mathbb{P}(0 \leftrightarrow r' | \tilde{\mathcal{F}}_r)}{\mathbb{P}(0 \leftrightarrow r')} \xrightarrow[L^\infty]{\text{a.s.}} \tilde{M}_r,$$

by the very definition of \tilde{M}_r , a random variable on the finite space B_r . Thus, for fixed r , the random variables $\mathbb{E}(M_{r'} | \tilde{\mathcal{F}}_r)$ are uniformly bounded in r' , and

$$\int_a^b \mathbb{E}(M_{r'}(\omega_s) | \tilde{\mathcal{F}}_r(\omega_s)) ds \xrightarrow[L^\infty]{\text{a.s.}} \int_a^b \tilde{M}_r(\omega_s) ds = \tilde{\mu}_r[a, b].$$

On the other hand, for random variables, convergence in L^2 is stronger than in L^1 , hence the L^2 -convergence (2.6) of the unconditional $\mu_{r'}[a, b]$ implies

$$\int_a^b \mathbb{E}\left(M_{r'}(\omega_s) \mid \tilde{\mathcal{F}}_r(\omega_s)\right) ds = \mathbb{E}\left(\mu_{r'}[a, b] \mid \tilde{\mathcal{F}}_r[a, b]\right) \xrightarrow[L^2]{\text{a.s.}} \mathbb{E}\left(\mu[a, b] \mid \tilde{\mathcal{F}}_r[a, b]\right).$$

One sequence can have only one a.s. limit, hence,

$$\mathbb{E}\left(\mu[a, b] \mid \tilde{\mathcal{F}}_r[a, b]\right) = \tilde{\mu}_r[a, b] \quad \text{almost surely.} \quad (2.7)$$

As $r \rightarrow \infty$, $\tilde{\mathcal{F}}_r[a, b]$ converges to the full sigma-algebra, hence the left-hand side of (2.7) converges a.s. to $\mu[a, b]$ by Lévy's zero-one law, while the right-hand side converges to $\tilde{\mu}[a, b]$, by (2.6). The two limits coincide a.s., thus the proof is complete. \square

Let us collect some basic properties of the dynamical percolation process, the exceptional set, and the associated local time.

Lemma 2.2 (Ergodicity). *The dynamical percolation process ω on the infinite lattice (in particular, the local time $\mu = \mu(\omega)$) is ergodic with respect to time shifts.*

Proof. For any dynamic event \mathcal{A} and any $\epsilon > 0$, there exists a radius $r \in \mathbb{N}$, a time $T > 0$, and an event $\mathcal{A}_{r,T}$ measurable with respect to $\omega^{B_r}(-T, T)$ such that $\mathbb{P}(\mathcal{A} \triangle \mathcal{A}_{r,T}) < \epsilon$. Now, by the ergodicity of dynamical percolation in B_r (a Markov chain on a finite state space), there exists $t = t(r, T)$ such that $|\mathbb{P}(\mathcal{A}_{r,T} \cap (\mathcal{A}_{r,T} + t)) - \mathbb{P}(\mathcal{A}_{r,T})^2| < \epsilon$, where $\mathcal{A}_{r,T} + t$ represents the event $\mathcal{A}_{r,T}$ evaluated for the dynamical configuration shifted back by time t . Now, if \mathcal{A} is invariant under time shifts, then $|\mathbb{P}(\mathcal{A} \cap (\mathcal{A} + t)) - \mathbb{P}(\mathcal{A}_{r,T} \cap (\mathcal{A}_{r,T} + t))| \leq \mathbb{P}(\mathcal{A} \triangle (\mathcal{A}_{r,T} \cap (\mathcal{A}_{r,T} + t))) < 2\epsilon$. Altogether, $|\mathbb{P}(\mathcal{A}) - \mathbb{P}(\mathcal{A})^2| < 2\epsilon + \epsilon + \epsilon^2$. This holds for any $\epsilon > 0$, hence $\mathbb{P}(\mathcal{A}) \in \{0, 1\}$. \square

Lemma 2.3 (Perfectness). *Almost surely, the set \mathcal{E} of exceptional times*

- (i) *is disjoint from the set of times at which the status of a hexagon is updated;*
- (ii) *is topologically closed;*
- (iii) *has no isolated points.*

Proof. Parts (i) and (ii) are proved in [HägPS97, Lemma 3.2]. Part (iii) is proved in [HägPS97, Lemma 3.4] and the remark following it. \square

Lemma 2.4 (No atoms). *There are almost surely no atoms in any of the measures μ_r , $\tilde{\mu}_r$ or $\mu = \tilde{\mu}$.*

Proof. Fix any large $n \in \mathbb{N}$, and cover the interval $[0, 1]$ by the intervals $I_i := [\frac{i}{2n}, \frac{i}{2n} + \frac{1}{n}]$, $i = 0, 1, \dots, 2n - 2$. By (2.5) and Chebyshev's inequality, for any $c > 0$ and any index i , we have $\mathbb{P}(\mu_r(I_i) > c) \leq n^{-1-\delta+o(1)}$ as $n \rightarrow \infty$, uniformly in r . By a union bound, $\mathbb{P}(\exists i : \mu_r(I_i) > c) \leq n^{-\delta+o(1)} \rightarrow 0$, which implies the claim for μ_r . Then (2.4) implies it for $\tilde{\mu}_r$, and (2.6) implies it for $\mu = \tilde{\mu}$. \square

It is clear that $\mu = \tilde{\mu}$ is supported inside \mathcal{E} . The following statement is very natural, but it seems hard to prove:

Conjecture 2.5. *The support of the local time measure μ is almost surely the entire exceptional time set \mathcal{E} .*

The conjecture cannot fail by much: the Hausdorff dimension of $\text{supp } \mu$ is the same as the dimension of \mathcal{E} , namely $31/36$. The reason is that the proof of the lower bound in [GPS10] (just like in [SchSt10]) uses the approximate local time measures μ_r and a version of the Mass Distribution Principle; hence this bound is really a lower bound on $\dim_H(\text{supp } \mu)$. The next lemma, which will be of use later, and which provides a little further evidence for the conjecture.

Lemma 2.6. *For any $\epsilon > 0$, let μ_ϵ denote $\mu_R[0, \epsilon]$ or the limit $\mu[0, \epsilon]$. Then there is an absolute constant $C < \infty$ such that*

$$\mathbb{E}(\mu_\epsilon^2 \mid \mu_\epsilon > 0) \leq C \mathbb{E}(\mu_\epsilon \mid \mu_\epsilon > 0)^2, \quad (2.8)$$

and another such constant $c > 0$ such that

$$\mathbb{P}(\mu_\epsilon > c \mathbb{E} \mu_\epsilon \mid \mathcal{E} \cap [0, \epsilon] \neq \emptyset) > c. \quad (2.9)$$

Proof. The left-hand side of (2.8) equals $\mathbb{E}(\mu_\epsilon^2)/\mathbb{P}(\mu_\epsilon > 0)$, while the right-hand side equals $\mathbb{E}(\mu_\epsilon)^2/\mathbb{P}(\mu_\epsilon > 0)^2 = \epsilon^2/\mathbb{P}(\mu_\epsilon > 0)^2$. Hence, we need to show that

$$\mathbb{E}(\mu_\epsilon^2) \leq C \frac{\epsilon^2}{\mathbb{P}(\mu_\epsilon > 0)}.$$

By a usual coupling between dynamical and near-critical percolation, in which dynamical updates lead always to the opening of hexagons in the latter case, we have

$$\begin{aligned} \mathbb{P}(\mu_\epsilon > 0) &\leq \mathbb{P}(\mathcal{E} \cap [0, \epsilon] \neq \emptyset) \leq \mathbb{P}_{p_c + O(\epsilon)}(0 \longleftrightarrow \infty) \\ &= O(1) \alpha_1(\rho(1/\epsilon)), \end{aligned} \quad (2.10)$$

by Kesten's scaling relation (1.10). On the other hand, taking the double integral of (1.15) over $s, t \in [0, \epsilon]$, we claim that

$$\mathbb{E}(\mu_\epsilon^2) \leq O(1) \frac{\epsilon^2}{\alpha_1(\rho(1/\epsilon))}, \quad (2.11)$$

which finishes the proof of (2.8). To verify (2.11), set $R = \rho(1/\epsilon)$ and $A_i = [C^i R, C^{i+1} R]$, $i \in \mathbb{N}$, where $C > 0$ is a large constant to be specified shortly. For $i \in \mathbb{N}$, write

$$B_i = \left\{ (s, t) \in [0, \epsilon]^2 : \rho(|s - t|^{-1}) \in A_i \right\},$$

so that, with μ_ϵ denoting $\mu_R[0, \epsilon]$,

$$\mathbb{E}(\mu_\epsilon^2) = \int_{[0, \epsilon]^2} \frac{\mathbb{P}\left(\mathbb{1}\{0 \xleftrightarrow{\omega_s} R\} \mathbb{1}\{0 \xleftrightarrow{\omega_t} R\}\right)}{\mathbb{P}(0 \longleftrightarrow R)^2} ds dt \leq O(1) \sum_{i \geq 0} \phi_i,$$

with $\phi_i = \int_{B_i} \frac{1}{\alpha_1(\rho(1/|t-s|))} ds dt$; the latter inequality is due to (1.15).

Note that $\rho(r)$ is a non-strictly increasing function. By (1.4), there exists an absolute constant $K > 0$ such that $s^2 \alpha_4(s) < (Ks)^2 \alpha_4(Ks)$ for all $s \in \mathbb{Z}^+$, hence $\rho(s^2 \alpha_4(s)) \in (s/K, s]$ for all $s \in \mathbb{Z}^+$. This implies that

$$\rho^{-1}(A_i) \subseteq [(C^i R)^2 \alpha_4(C^i R), (C^{i+1} K R)^2 \alpha_4(C^{i+1} K R)]. \quad (2.12)$$

If C is large enough, so that $(CR)^2 \alpha_4(CR) > 2(KR)^2 \alpha_4(KR)$, then $(1/\epsilon, 2/\epsilon) \subseteq \rho^{-1}(A_0)$, hence the Lebesgue measure of B_0 is at least $\epsilon^2/2$. Therefore,

$$\phi_0 = \int_{B_0} \frac{1}{\alpha_1(\rho(1/|t-s|))} ds dt \geq \frac{\epsilon^2}{2} \alpha_1(\rho(1/\epsilon))^{-1}.$$

On the other hand, for $i \geq 1$, using (2.12),

$$\phi_i = \int_{B_i} \frac{1}{\alpha_1(\rho(1/|t-s|))} ds dt \leq 2\epsilon C^{-2i} R^{-2} \alpha_4(C^i R)^{-1} \alpha_1(C^{i+1} R)^{-1}.$$

Thus,

$$\begin{aligned} \frac{\phi_i}{\phi_0} &\leq 4\epsilon^{-1} C^{-2i} R^{-2} \frac{\alpha_1(R)}{\alpha_4(C^i R) \alpha_1(C^{i+1} R)} \\ &\leq \frac{4C^{-2i}}{\alpha_4(R, C^i R) \alpha_1(R, C^{i+1} R)}, \end{aligned}$$

where in the second inequality we used that $\epsilon^{-1} \leq R^2 \alpha_4(R)$. Now, [GPS10, Appendix] says that the sum of the 1-arm and 4-arm exponents is strictly less than 2 — properly interpreted in the case of \mathbb{Z}^2 where these exponents are not known to exist. That is, there exists some $c \in (0, 1)$ such that $\phi_i/\phi_0 \leq O(1) c^i$ for all $i \geq 1$. Thus,

$$\int_{[0, \epsilon]^2} \frac{1}{\alpha_1(\rho(1/|t-s|))} ds dt \leq O(1) \int_{A_0} \frac{1}{\alpha_1(\rho(1/|t-s|))} ds dt \leq O(1) \epsilon^2 \alpha_1(\rho(1/\epsilon))^{-1},$$

and we have confirmed (2.11).

By the Paley-Zygmund second moment inequality (a simple consequence of Cauchy-Schwarz; see, e.g., [LyP11, Section 5.5]), the above computations show that

$$\mathbb{P}(\mu_\epsilon > 0) \geq \frac{(\mathbb{E}\mu_\epsilon)^2}{\mathbb{E}(\mu_\epsilon^2)} \geq c_1 \alpha_1(\rho(1/\epsilon)),$$

matching the upper bound (2.10) up to a constant factor. Therefore,

$$\mathbb{P}(\mu_\epsilon > 0 \mid \mathcal{E} \cap [0, \epsilon] \neq \emptyset) > c_2 > 0.$$

On the other hand, again by the Paley-Zygmund inequality, (2.8) implies that

$$\mathbb{P}(\mu_\epsilon > c_3 \mathbb{E}\mu_\epsilon \mid \mu_\epsilon > 0) > c_3 > 0,$$

for some $c_3 > 0$. Combining the last two displayed inequalities proves (2.9). \square

We conclude this section with a natural question:

Question 2.7. *Is the local time μ the 31/36-dimensional Minkowski content of the set \mathcal{E} ? Is μ the Hausdorff measure of \mathcal{E} for some Hausdorff gauge function?*

3 Finding the Incipient Infinite Cluster

Given the description of the local time measure using (2.2), it is natural to guess that the infinite cluster at a “typical” exceptional time (typical with respect to μ) has the law of IIC. The first exceptional time having been discredited as a candidate for the IIC by Theorem 1.3, we now prove Theorems 1.7 and 1.8, thereby verifying what may be the simplest relationship between exceptional times and the IIC.

Unsurprisingly, the proofs go through the finite approximations, about which we provide a further definition.

Definition 3.1. *Let IIC_r denote the law on percolation configurations in B_r given by $\mathbb{P}_{p_c}(\cdot \mid 0 \leftrightarrow r)$.*

Note that $M_r(\omega)$ is the Radon-Nikodym derivative $d\text{IIC}_r/d\mathbb{P}$, while $\tilde{M}_r(\omega)$ is the Radon-Nikodym derivative $d\text{IIC}^{B_r}/d\mathbb{P}$, where $\mathbb{P} = \mathbb{P}_{p_c}$ is critical percolation. Since both IIC_r and IIC^{B_r} converge to IIC as $r \rightarrow \infty$, it will turn out that we could use either M_r and μ_r or \tilde{M}_r and $\tilde{\mu}_r$ in our proofs (with minor modifications); we have chosen to use the former. The finite versions will be slightly stronger than the infinite ones, in that they identify not only a moment where we get the IIC_r , but also an equality of entire processes. We will use this stronger, dynamic version in Section 4.

Lemma 3.2 (Finite r quenched sampling). *Let $\{\omega(s) : s \in [0, \infty)\}$ be dynamical percolation in B_r . Let $\chi_{r,T} \in \mathbb{R}$ be a random time sampled from $\mu_r/\mu_r[0, T]$, defined only when $\mu_r[0, T] > 0$. Then $\{\omega(\chi_{r,T} + s) : s \in [0, \infty)\}$, in the limit $T \rightarrow \infty$ for almost all ω , is equal in law to standard dynamical percolation started from IIC_r at time zero.*

Lemma 3.3 (Finite r annealed sampling).

- (a) *Let $\{\omega^*(s) : s \in [0, \infty)\}$ be dynamical percolation in B_r size-biased by $\mu_r[0, T]$, and $\chi_{r,T}^* \in \mathbb{R}$ be a random time with law $\mu_r/\mu_r[0, T]$ for $\mu_r = \mu_r(\omega^*)$. Then the process $\{\omega^*(\chi_{r,T}^* + s) : s \in [0, \infty)\}$ is equal in law to standard dynamical percolation started from IIC_r at time zero.*

- (b) The Palm version (ω^*, Π_r^*) of the process $(\omega, \Pi_{\mu_r}(\omega))$ in B_r is standard dynamical percolation started from $\Pi_{\mathcal{C}_r}$ at time zero. A somewhat concrete way to realize the Palm version is **Liggett's extra head construction** [Lig02], see Figure 3.1.

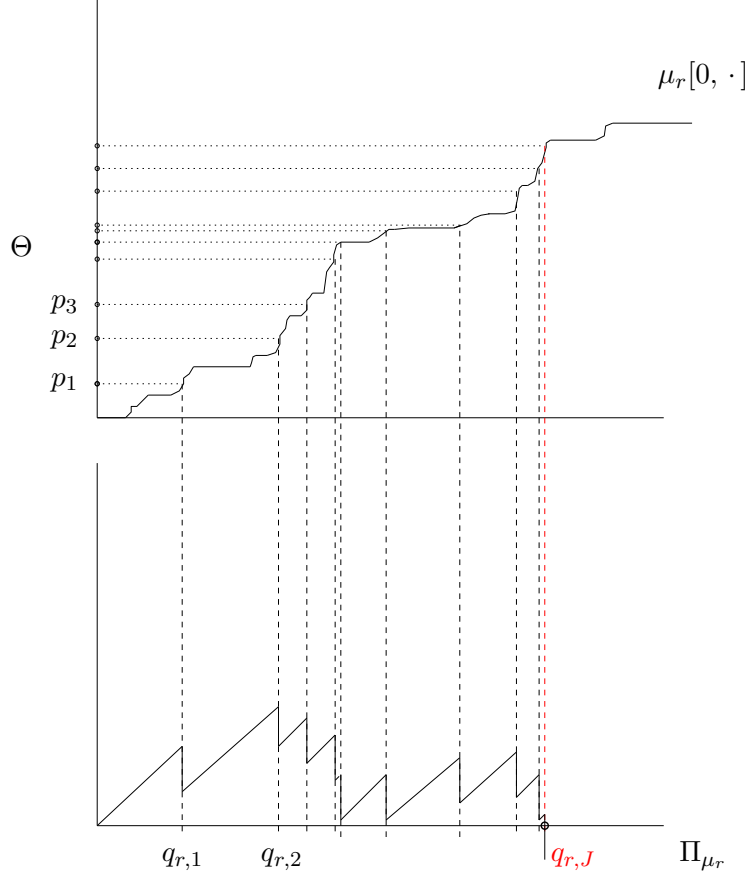


Figure 3.1: Depicting Liggett's extra head construction.

Let $\{p_i \in [0, \infty) : i \in \mathbb{N}\}$ enumerate a Poisson point process Θ with intensity measure Lebesgue on $[0, \infty)$, and set $q_{r,i} = \inf \{t > 0 : \mu_r[0, t] > p_i\}$. Clearly, $\Pi_{\mu_r} := \{q_{r,i} : i \in \mathbb{N}\}$ is a Poisson point process with intensity μ_r . Set $m_r := \mathbb{E}\mu_r[0, 1]$. Now let $J \in \mathbb{N}$ be the first integer with $|\Pi_{\mu_r} \cap [0, m_r^{-1}J]| > J$. Then shifting back time by $q_{r,J}$ gives the Palm version of (ω, Π_{μ_r}) .

Proof of Lemma 3.2. The stationary measure of the chain restricted to the set $\{0 \leftrightarrow r\}$ is $\Pi_{\mathcal{C}_r}$. Hence the lemma is straightforward from ω being ergodic (see Lemma 2.2) and from the strong Markov property. \square

Proof of Lemma 3.3. The Palm version of a stationary process (ω, ξ) on \mathbb{R} , where ξ is a random measure, is defined in [Kal02, Chapter 11] as follows. For any Borel set $B \subset \mathbb{R}$ of positive Lebesgue measure, and any nonnegative measurable function f on configurations (ω, ξ) , consider $\xi_f(B) := \int_B f(\theta_s(\omega, \xi)) \xi(ds)$, where θ_s is the shift by $-s$. Then the Palm

version is the law defined by $Q_{\omega,\xi}[f] := \mathbb{E}\xi_f(B)/\mathbb{E}\xi(B)$. It is not hard to show that this does not depend on B .

If we take $\xi = \mu_r$ and $B = [0, T]$, then this construction specializes to the process defined in part (a). Since we know from Lemma 2.2 that (ω, μ_r) is ergodic, we can apply [Kal02, Theorem 11.6], saying that this Palm version equals the process defined in Lemma 3.2, hence the claim of part (a) follows from that lemma. (Note that it is pretty clear why the ergodic quenched limit leads to a size-biased finite average: each dynamic configuration of a finite time interval appears in the ergodic quenched limit with a frequency proportional to its probability.)

Now take $\xi = \Pi_{\mu_r}$, and the Borel sets $B_\epsilon := (-\epsilon, \epsilon)$. [Kal02, Theorem 11.5] says that the Palm version of (ω, Π_{μ_r}) is the same as conditioning on $|B_\epsilon \cap \Pi_{\mu_r}| \geq 1$ or on $|B_\epsilon \cap \Pi_{\mu_r}| = 1$, then taking the limit $\epsilon \rightarrow 0$. This is the most common form of taking the ‘‘Palm version of a point process’’. Note that for the equivalence of definitions here, we need that μ_r does not have atoms (by Lemma 2.4), hence Π_{μ_r} is a simple point process. Furthermore, since μ_r has a density, M_r , we can again give a two-sentence intuitive explanation for this Palm conditioning (at least for the static time-zero version): for any static percolation configuration ζ in B_r , we have

$$\begin{aligned} \mathbb{P}(B_\epsilon \cap \Pi_{\mu_r} \neq \emptyset \mid \omega(0)^{B_r} = \zeta, \mu_r) &= 1 - \exp\left(-\int_{-\epsilon}^{\epsilon} M_r(\omega_t) dt\right) \\ &\sim 2\epsilon M_r(\zeta) \quad \text{a.s. as } \epsilon \rightarrow 0, \end{aligned}$$

by the Lebesgue integration theorem and Fubini. Therefore, $M_r(\zeta)$ being the Radon-Nikodym derivative $d\Pi_{\mu_r}/d\mathbb{P}$, the Palm conditioning gives

$$\lim_{\epsilon \rightarrow 0} \mathbb{P}(\omega(0)^{B_r} = \zeta \mid B_\epsilon \cap \Pi_{\mu_r} \neq \emptyset) = \Pi_{\mu_r}(\zeta).$$

Since Π_{μ_r} is obtained from μ_r using independent stationary randomness, the ω^* marginal in the Palm version of (ω, Π_{μ_r}) is the same as in the Palm version of (ω, μ_r) , which we already described in part (a).

Finally, regarding Liggett’s extra head construction, [Lig02, Corollary 4.18] says that shifting back by $q_{r,J}$ as defined in part (b) produces the Palm version of Π_{μ_r} . Now we need to extend this result from the marginal Π_{μ_r} to (ω, Π_{μ_r}) . Note that Liggett’s construction gives a random of time shift $T_{J_{q,r}}$ that is measurable with respect to Π_{μ_r} , where each time shift T_x is a measure-preserving transformation on the space of dynamical configurations ω . Therefore, if \mathcal{A} and \mathcal{B} are arbitrary events for the Palm version $\Pi_{\mu_r}^*$, and $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are the events for $(\omega^*, \Pi_{\mu_r}^*)$ that project to \mathcal{A} and \mathcal{B} in the second coordinate, then

$$\frac{\mathbb{P}(T^{-1}(\mathcal{A}))}{\mathbb{P}(T^{-1}(\mathcal{B}))} = \frac{\mathbb{P}(T^{-1}(\tilde{\mathcal{A}}))}{\mathbb{P}(T^{-1}(\tilde{\mathcal{B}}))},$$

whenever the denominator on either side of this equation is positive. See Figure 3.2. This implies that the effect of T is the same as conditioning on $\{0 \in \Pi_{\mu_r}\}$ not only on Π_{μ_r} but also on (ω, Π_{μ_r}) , and we are done. \square

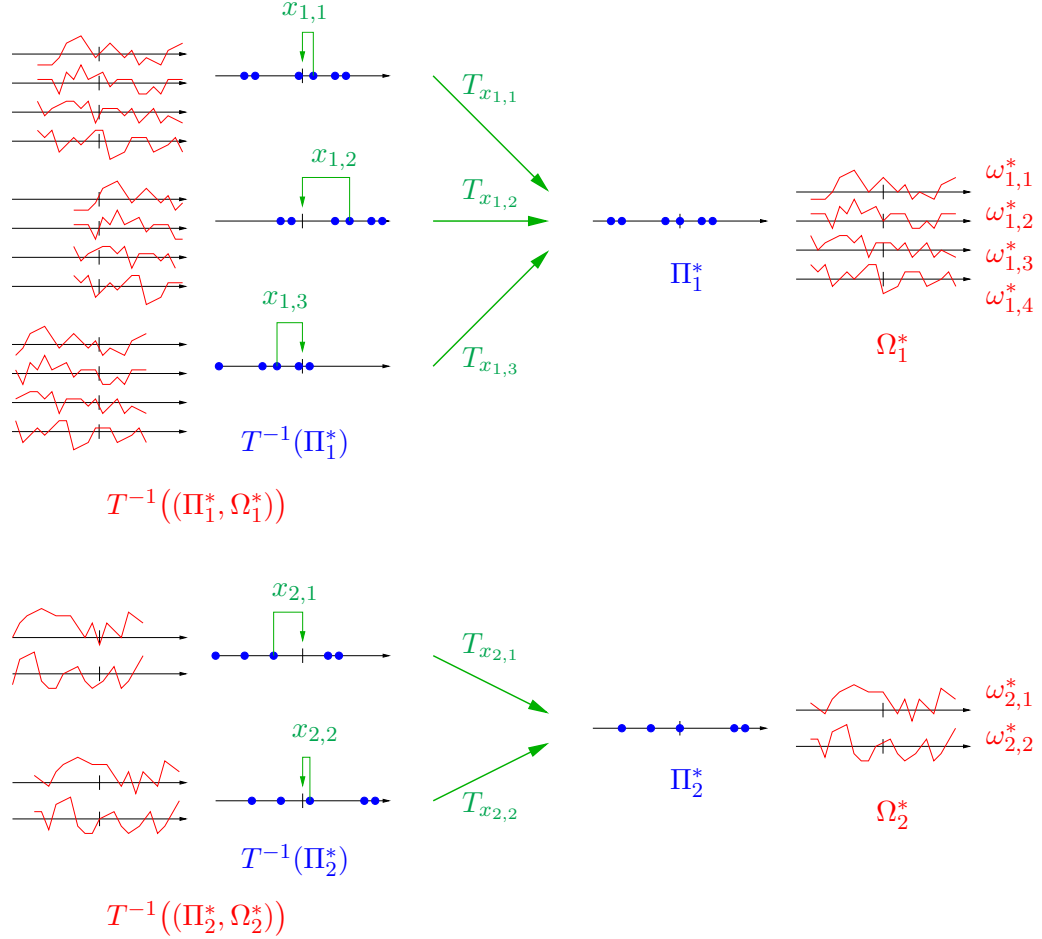


Figure 3.2: A schematic picture of the effect of Liggett's extra head time shift T on (Π_{μ_r}, ω) . Although different Palm point process realizations Π_1^* and Π_2^* may arise from a different amount of Palm dynamical percolation realizations $\Omega_1^* = \{\omega_{1,i}^* : i \in I_1\}$ and $\Omega_2^* = \{\omega_{2,i}^* : i \in I_2\}$, and the preimages $T^{-1}(\Pi_1^*)$ and $T^{-1}(\Pi_2^*)$ might also have different sizes, the ratio of these sizes coincide with the ratio for the preimages $T^{-1}((\Pi_1^*, \Omega_1^*))$ and $T^{-1}((\Pi_2^*, \Omega_2^*))$.

Proof of Theorem 1.7. We must argue that $\mu[0, T] > 0$ for all T sufficiently high, and also that, for each $r \in \mathbb{N}$, $\omega(\chi_T)|_{B_r}$ converges weakly as $T \rightarrow \infty$ to $\Pi|_{B_r}$.

For $n \in \mathbb{N}$, set $I_i^n = [i/n, (i+1)/n)$. For $R \in \mathbb{N}$, define $f_R^n : [0, \infty) \rightarrow [0, \infty)$ according to $f_R^n(x) = n\mu_R(I_i^n)$ if $x \in I_i^n$ for $i \in \mathbb{N}$. Similarly define $f_\infty^n : [0, \infty) \rightarrow [0, \infty)$ according to $f_\infty^n(x) = n\mu(I_i^n)$ if $x \in I_i^n$ for $i \in \mathbb{N}$. We now argue that, for each $\epsilon > 0$ and $n \in \mathbb{N}$,

there exists $R \in \mathbb{N}$ such that

$$\limsup_T \int_0^T \left| \frac{f_R^n(t)}{\int_0^T f_R^n(s)ds} - \frac{f_\infty^n(t)}{\int_0^T f_\infty^n(s)ds} \right| dt \leq \epsilon. \quad (3.1)$$

Note that this assertion allows us to construct couplings \mathbf{Q} of $\chi_{R,T}$ and χ_T as follows: for each $\epsilon > 0$ and $n \in \mathbb{N}$, there exists $R \in \mathbb{N}$ such that $\chi_{R,T}$ and χ_T are coupled under $\mathbf{Q} = \mathbf{Q}_R$ so that

$$\limsup_T \mathbf{Q}(|\chi_{R,T} - \chi_T| \geq 1/n) \leq \epsilon. \quad (3.2)$$

Note that (3.1) is a consequence of the next three assertions. First, for each $\epsilon > 0$, there exists $R \in \mathbb{N}$ such that

$$\limsup_T T^{-1} \int_0^T |f_R^n(t) - f_\infty^n(t)| dt \leq \epsilon, \quad \mathbb{P}\text{-almost surely}. \quad (3.3)$$

Second, for each $\epsilon > 0$, and for this same value of $R \in \mathbb{N}$,

$$\limsup_T T^{-1} \left| \int_0^T f_R^n(s)ds - \int_0^T f_\infty^n(s)ds \right| \leq \epsilon, \quad \mathbb{P}\text{-almost surely}. \quad (3.4)$$

Third,

$$\lim_T T^{-1} \int_0^T f_\infty^n(s)ds = 1, \quad \mathbb{P}\text{-almost surely}. \quad (3.5)$$

We now justify (3.3), (3.4) and (3.5).

To confirm (3.3), note that $\mathbb{E}(\int_0^{1/n} |f_R^n(t) - f_\infty^n(t)| dt) = \int |\mu(0, 1/n) - \mu_R(0, 1/n)| d\mathbb{P}$. We fix $R \in \mathbb{N}$ by Theorem 2.1 so that $\int |\mu(0, 1/n) - \mu_R(0, 1/n)| d\mathbb{P} \leq \epsilon/n$. Lemma 2.2 then provides (3.3).

Note that (3.4) is a trivial consequence of (3.3).

To show (3.5), note that, by definition, $\mathbb{E}(\mu_r[0, 1]) = 1$ for each $r \in \mathbb{N}$. Thus Theorem 2.1 implies that $\mathbb{E}(\mu[0, 1]) = 1$. Lemma 2.2 then implies that $\lim_T T^{-1} \mu(0, T) = 1$, \mathbb{P} -almost surely. This limit coincides with that in (3.5), which establishes this claim. Note that in this derivation we have confirmed that indeed $\mu(0, T) > 0$ for T sufficiently high, \mathbb{P} -almost surely.

We conclude the proof by arguing that, for each $\epsilon > 0$ and $r \in \mathbb{N}$, there exists $R \in \mathbb{N}$ such that

$$\liminf_T \mathbf{Q}(\omega(\chi_{R,T})|_{B_r} = \omega(\chi_T)|_{B_r}) \geq 1 - \epsilon. \quad (3.6)$$

This indeed suffices for Theorem 1.7: recall that we must argue that, for each $r \in \mathbb{N}$, $\omega(\chi_T)|_{B_r}$ converges weakly as $T \rightarrow \infty$ to $\Pi C|_{B_r}$. For each $R \in \mathbb{N}$, we know by Lemma 3.2 that the weak limit as $T \rightarrow \infty$ of $\omega(\chi_{R,T})$ equals ΠC_R . Fixing $\epsilon > 0$, and for given $r \in \mathbb{N}$, choose $R \geq r$ high enough that $\Pi C|_{B_r}$ and $\Pi C_R|_{B_r}$ have total variation distance at most ϵ . For large enough T then, the total variation distance between $\omega(\chi_{R,T})|_{B_r}$ and $\Pi C|_{B_r}$ is at most 2ϵ . (Note here that on the discrete topological space $\{0, 1\}^{B_r}$, convergence in law is

the same as in total variation distance.) On the other hand, by (3.6), $\omega(\chi_T)$ coincides with $\omega(\chi_{R,T})$ on B_r with \mathbf{Q} -probability at least $1 - \epsilon$ for all high enough T . Thus the total variation distance between $\omega(\chi_T)|_{B_r}$ and $\text{IIC}|_{B_r}$ becomes less than 3ϵ , and we are done.

It remains only to verify (3.6). In light of (3.2), it is enough to argue that, for given $\epsilon > 0$ and $R, r \in \mathbb{N}$ with $R \geq r$, there exists $n \in \mathbb{N}$ such that, for all T sufficiently high, the \mathbf{Q} -probability that a hexagon in B_r flips during $[\chi_{R,T} - 1/n, \chi_{R,T} + 1/n]$ is at most ϵ . However, by Lemma 3.2, the times of hexagon flips in B_r during $[\chi_{R,T} - 1/n, \chi_{R,T} + 1/n]$, shiftward backwards in time by $\chi_{R,T}$, converges weakly as $T \rightarrow \infty$ to a Poisson process of rate $|B_r|/2$ on $[-1/n, 1/n]$. Choosing $n \leq c_\epsilon r^{-2}$ thus gives the desired statement. \square

Proof of Theorem 1.8. Part (a) follows from Theorem 1.7 — by Lemma 2.2 and [Kal02, Theorem 11.6] — just as Lemma 3.3 followed from Lemma 3.2.

Part (b) follows from Lemma 3.3(b) and the next two lemmas. \square

Lemma 3.4. *If $\tau \in \mathcal{E}$ is an exceptional time, and $\tau_n \rightarrow \tau$, then, for any $r > 0$, we have $\omega(\tau_n)|_{B_r} = \omega(\tau)|_{B_r}$ for all sufficiently high n .*

Proof. By Lemma 2.3(i), there is an open interval I which contains the exceptional time τ such that, for $t \in I$, $\omega(t)|_{B_r} = \omega(\tau)|_{B_r}$; hence $\tau_n \rightarrow \tau$ implies the lemma. \square

Lemma 3.5. *For the times $q_{r,J}$ defined in Lemma 3.3 (b), the limit $q_J := \lim_{r \rightarrow \infty} q_{r,J}$ exists almost surely.*

Proof. If $f_n : [0, \infty) \rightarrow [0, \infty)$ is a sequence of non-decreasing functions converging pointwise on $[0, \infty)$ to a function $f : [0, \infty) \rightarrow [0, \infty)$, and we write

$$f_n^{-1}(x) = \inf \{t > 0 : f_n(t) > x\},$$

then, whenever $x \in [0, \infty)$ is a point of increase of $f : [0, \infty) \rightarrow [0, \infty)$, we have that $\lim_{n \rightarrow \infty} f_n^{-1}(x) = f^{-1}(x)$. The following thus suffices for Lemma 3.5.

Lemma 3.6. *For any $p \in \Theta$, $t_\rho := \inf \{s > 0 : \mu(0, s) > \rho\}$ is almost surely a point of increase of $\mu(0, \cdot)$.*

Proof. Note that the set of $\rho \in (0, \infty)$ for which t_ρ is a point of increase of $\mu(0, \cdot)$ is given by $\mathbb{R} \setminus \mu(0, Q)$, with $\mu(0, Q) = \{\mu(0, q) : q \in Q\}$, where Q is the collection of left-hand endpoints of intervals comprising $\text{supp}(\mu)^c$. Note that Q is countable, and, thus, is so $\mu(0, Q)$. Thus, $\Theta \cap \mu(0, Q) = \emptyset$ a.s., because Θ is independent of $\mu(0, Q)$. \square

4 FETIC is not IIC

In this section, we prove Theorem 1.3.

4.1 The skeleton of the argument

Definition 4.1. Let ω be a sample of dynamical percolation in the R -ball B_R . Write \mathcal{E}_R for the set of times such that $0 \leftrightarrow R$. Let $\text{FET}_R = \inf \{t \geq 0 : 0 \xleftrightarrow{\omega_t} R\}$, and let FETIC_R be the law of ω_{FET_R} conditioned on $\text{FET}_R > 0$. (Since $\mathcal{C}_0(\omega_0)$ is almost surely finite, it takes positive time for the first bit on its boundary to change, and hence the event $\text{FET}_R > 0$ is the same as $0 \nleftrightarrow R$ in ω_0 , which is almost surely satisfied for large enough R .)

These finite approximations will be very useful. On the one hand, IIC_R is the law of the configuration at a typical point of \mathcal{E}_R , as we saw in Lemmas 3.2 and 3.3. On the other hand, by [HamMP12, Lemma 4.5], we have that $\text{FET}_R \rightarrow \text{FET}$ almost surely as $R \rightarrow \infty$; hence FETIC_R converges to FETIC in law (by Lemma 3.4).

There is a natural line of attack if we want to distinguish IIC_R from FETIC_R . Let us call the left-isolated points of \mathcal{E}_R **arrivals**, and the right-isolated points **departures**. As we will see, the law of a *typical* arrival configuration can be easily obtained from IIC_R (and will be denoted by IIC'_R): we get it by size-biasing with respect to the number of pivotal hexagons for the event $\{0 \leftrightarrow R\}$. This is different from IIC_R , but not by much: it can be shown (though we will not do so) that its weak limit as $R \rightarrow \infty$ coincides with IIC . However, FETIC_R is not given by a typical arrival: as usual when waiting for the first arrival of a stationary point process, the time between FET_R and the last departure before it (somewhere in the negative half-line) is a size-biased sample of the typical reconnection time between departures and arrivals, and if an arrival configuration typically occurs at the end of longer disconnection intervals, then it is more likely to appear in FETIC_R . Since it is harder to think about dynamical percolation ending at a certain configuration than about starting it at such a configuration, our strategy to understand FETIC_R will be to reverse time, start dynamical percolation from certain typical IIC'_R configurations, condition on immediate termination of $\{0 \leftrightarrow R\}$, and then estimate the expected time of reconnection. If we can exhibit two events at time zero that have the same positive probability under the limit measure IIC , but for which the expected reconnection times differ, then these events will turn out to have different probabilities under FETIC , and we will be done.

Roughly, of these two events under IIC'_R , the first will be that the configuration looks “normal” in a bounded neighbourhood of 0, while the second will be that the configuration is “thinner” in the same neighbourhood. (We will in fact define a thinning procedure on normal static configurations satisfying $0 \leftrightarrow R$, changing the configuration in a bounded neighbourhood of 0.) A thinner configuration falls apart more easily, and hence reconnects to distance R with more difficulty; and so, one may expect that such a configuration is more probable under FETIC_R than is a normal configuration, which is to say, FETIC_R is thinner than IIC_R . This is certainly the case if the thin configuration is, say, given by a single straight line segment of open hexagons from 0 to ∂B_R , with all other hexagons in B_R being closed. However, this R -dependent configuration has a vanishing probability in the limit measure

IIC; therefore, while the imbalance in probability of this configuration distinguishes FETIC_R from IIC_R , a distinction between FETIC and IIC cannot be deduced. This is why we want to require the configuration to be thin only in a bounded neighbourhood of 0. However, the main difficulty now is that normal reconnection times are very short if R is large, and that, with high probability, the configuration is entirely static in a bounded neighbourhood of the origin; hence it is not clear that our thinning will have a noticeable effect on the reconnection time. The solution will be that the expected reconnection time, though tiny, turns out to be dominated by times that are macroscopically large (independently of R): large enough that if the configuration close to 0 is thin then it does indeed start falling apart, making expected reconnection time noticeably larger when the thinning procedure has been applied. To argue this, we will need the result from [HamMP12] that FET has finite expectation (in fact, an exponential tail): this will tell us that the normal reconnection time is well behaved, making it possible to prove that, in expectation, it is strictly dominated by the reconnection time of thinned configurations.

In this introductory subsection, we first explain the time-reversal and the size-biasing effects determining the relationship between IIC_R and FETIC_R , then define the thinning procedure, and will finally show that a noticeable difference between expected reconnection times indeed implies that FETIC and IIC are different. In the subsequent subsections, we will prove that there is such a difference.

Recall from the above discussion that, in standard càdlàg dynamical percolation, a time $t \in \mathcal{E}_R$ for which there exists $\epsilon > 0$ such that $[t - \epsilon, t) \cap \mathcal{E}_R = \emptyset$ is called an arrival. Write \mathcal{A}_R for the set of arrivals. Furthermore, for a static percolation configuration ζ in B_R that satisfies $0 \leftrightarrow R$, denote by $\text{Piv} = \text{Piv}_{0 \leftrightarrow R}(\zeta)$ the set of hexagons in B_R that are pivotal in the configuration ζ for $0 \leftrightarrow R$, and recall that IIC'_R denotes the law on configurations in B_R whose Radon-Nikodym derivative with respect to IIC_R is given by $|\text{Piv}|$ up to normalization.

Lemma 4.2. *The following three definitions for the process $\mathbb{P}(\cdot \mid 0 \in \mathcal{A}_R)$ are equivalent:*

- (i) *consider dynamical percolation in B_R conditionally on the event $\{0 \leftrightarrow R\}$ occurring at time 0 but not at time $-\epsilon$, and take the weak limit as $\epsilon \downarrow 0$;*
- (ii) *for large $T > 0$, pick uniformly an element $\tau \in \mathcal{A}_R \cap [0, T]$, consider the shifted dynamical percolation configuration $\{\omega_{t-\tau} : t \in \mathbb{R}\}$, and take the weak limit as $T \rightarrow \infty$ (conditionally on ω , or averaged);*
- (iii) *let ω_0 be distributed according to IIC'_R , choose uniformly an element $S \in \text{Piv}(\omega_0)$, obtain the configuration ω_{0-} by closing the hexagon S , and let the rest of the evolution $\{\omega_t : t \in \mathbb{R}\}$ be given by càdlàg dynamical percolation updates independently of the values of ω_0 and S .*

Proof. While the weak limits in (i) and (ii) might not exist a priori, the definition of (iii) is clearly well formulated. We first prove the equivalence of (i) and (iii), implying the existence

of the weak limit in (i), in particular. It is enough to show that, for all configurations ζ in B_R such that $0 \leftrightarrow R$,

$$\lim_{\epsilon \rightarrow 0} \frac{d\mathbb{P}(\cdot \mid 0 \in \mathcal{E}_R, -\epsilon \notin \mathcal{E}_R)}{d\mathbb{IIC}_R}(\zeta) = Z_1^{-1} |\text{Piv}_{0 \leftrightarrow R}(\zeta)|, \quad (4.1)$$

where $Z_1 \in (0, \infty)$ is a normalization.

Given a configuration ζ such that $0 \leftrightarrow R$, let $p_\epsilon(\zeta)$ be the probability that dynamical percolation given $\omega_0 = \zeta$ satisfies $0 \not\leftrightarrow R$ at time $-\epsilon$. If ϵ is tiny (depending on R), then the probability of having at least two hexagons flipping in the time interval $(-\epsilon, 0)$ is much less than the probability of any specific hexagon flip. Therefore, $\lim_{\epsilon \rightarrow 0} p_\epsilon(\zeta)/\epsilon = |\text{Piv}_{0 \leftrightarrow R}(\zeta)|$, which implies (4.1).

To prove the equivalence of (ii) and (iii), let us reformulate the T -dependent law defined in (ii) as taking uniformly one from all pairs of configurations $(\omega_{t-}, \omega_t) \in \mathcal{E}_R^c \times \mathcal{E}_R$, with $t \in [0, T]$, and then running dynamical percolation in the two directions from here. By the ergodicity of $\{\omega_t : t \in \mathbb{R}\}$ (Lemma 2.2), the weak limit of this law is the same as taking a pair of static configurations (ζ_1, ζ_2) that differ only in one hexagon such that $0 \leftrightarrow R$ in ζ_2 but not in ζ_1 to start the dynamics. This is clearly the same as the law defined in (iii).

The equivalence of (i) and (ii) follows from the above two equivalences; or, just like in Lemma 3.3, we can also quote [Kal02, Theorem 11.6] on the equivalent definitions of the Palm version of the process (ω, \mathcal{A}_R) . \square

Now, as we promised, in order to understand the effect of waiting for the first exceptional time on the distribution of the configuration at that time, we time-reverse the dynamics, started from typical arrival times:

Definition 4.3. Let \mathbb{P}_{norm} denote the time-reversal of $\mathbb{P}(\cdot \mid 0 \in \mathcal{A}_R)$ (i.e., $t \mapsto -t$ for all $t \in \mathbb{R}$). More explicitly, it is the càglàd (left-continuous with right limits) Markov process given as follows. Under \mathbb{P}_{norm} , the distribution of ω_0 is \mathbb{IIC}'_R . Given ω_0 , a uniform element $S \in \text{Piv}$ is selected, with the configuration ω_{0+} being set equal to ω_0 modified by closing the hexagon S . The rest of the evolution of $\{\omega_t : t \in \mathbb{R}\}$ is given by càglàd dynamical percolation updates independently of the values of ω_0 and S .

Lemma 4.4. Under the law \mathbb{P}_{norm} , recall that $0 \leftrightarrow R$ is satisfied by ω_0 but not by ω_{0+} ; let the reconnection time $N \in (0, \infty)$ be given by $N = \inf \{t > 0 : 0 \xleftrightarrow{\omega_t} R\}$. For each static B_R configuration ζ , we have that

$$\frac{d\text{FETIC}_R}{d\mathbb{IIC}_R}(\zeta) = Z^{-1} \mathbb{E}_{\text{norm}}(N \mid \omega_0 = \zeta) |\text{Piv}_{0 \leftrightarrow R}|,$$

where $Z \in (0, \infty)$ is a normalization.

Proof. We claim that

$$\frac{d\text{FETIC}_R}{d\mathbb{P}(\cdot \mid 0 \in \mathcal{A}_R)}(\zeta) = Z_2^{-1} \mathbb{E}_{\text{norm}}(N \mid \omega_0 = \zeta), \quad (4.2)$$

where $Z_2 \in (0, \infty)$ is another normalization. From (4.1) and (4.2) follows the statement of the lemma.

To prove (4.2), let $\phi : \mathcal{E}_R^c \rightarrow \mathcal{A}_R$ associate to each moment of disconnection $0 \nleftrightarrow R$ in càdlàg dynamical percolation the first connection time to its right (which is necessarily an arrival). Condition the process on $\mathcal{E}_R^c \cap [-n, 0] \neq \emptyset$ and pick a random time χ whose conditional law is given by normalized Lebesgue measure on $\mathcal{E}_R^c \cap [-n, 0]$; note that FETIC_R is the weak limit as $n \rightarrow \infty$ of $\omega_{\phi(\chi)}$. Note that, in this weak limit, the probability that $\omega_{\phi(\chi)}$ is a given static configuration ζ (for which $0 \leftrightarrow R$) is proportional to the mean length of an interval in \mathcal{E}_R^c at whose right-hand endpoint the configuration is ζ . Thus we obtain (4.2). \square

Here is a straightforward variant of (4.2). For any non-negative random variable X of finite mean, \widehat{X} will denote the size-biased version; i.e., $\mathbb{P}(\widehat{X} \geq t) = \mathbb{E}(X)^{-1} \mathbb{E}(X \mathbb{1}_{X \geq t})$.

Lemma 4.5. *Let \widehat{N} be the size-biased version of the reconnection time N under the law \mathbb{P}_{norm} , and let U be an independent $\text{Unif}[0, 1]$ random variable. Then $\widehat{N}U$ has the distribution of FET_R .*

The following useful fact was proved in [HamMP12].

Lemma 4.6. *In dynamical percolation we have*

$$\mathbb{P}(\text{FET}_R > t) \leq \exp\{-ct\}$$

for all $t > 0$, where $c > 0$ may be chosen uniformly in $R \in \mathbb{N}$.

Note that the preceding two lemmas imply that $\mathbb{P}(\widehat{N} > t) \leq \exp\{-ct\}$, uniformly in R . In particular, this random variable has finite moments: for each $k \in \mathbb{N}$, $\mathbb{E}(\widehat{N}^k) = \mathbb{E}(N^{k+1})/\mathbb{E}N < \infty$, again uniformly in R .

We now introduce the thinning procedure which is central to our technique for showing that FETIC differs from IIC .

Definition 4.7. *A circuit Γ is a finite self-avoiding path of hexagons such that for no vertex in the hexagonal lattice are all three of the neighbouring hexagons visited by Γ and such that $\mathcal{H} \setminus \Gamma$ has exactly two connected components: a finite one, denoted by $\text{Int}(\Gamma)$, and an infinite one. Note that a partial order on circuits Γ is provided by containment of the enclosed regions $\text{Int}(\Gamma)$.*

Let ζ be a percolation configuration in B_R such that $0 \leftrightarrow R$. Note that if some ζ -open circuit Γ satisfies $B_r \subseteq \text{Int}(\Gamma)$, then there is a unique ζ -open circuit which encloses B_r and is minimal in the partial order among such circuits. If ζ is such that this circuit exists, we label the circuit by Γ_r .

Definition 4.8. Recall the exponent $\eta \in (0, 1)$ from (1.4), and fix $\epsilon > 0$ small enough that $(1 + 2\epsilon)(1 - \eta) < 1$. Now assume that r satisfies $r^{2(1+2\epsilon)}\alpha_4(r^{1+2\epsilon}) < r/2$, which holds for all large enough r , by (1.5). Let $R \in \mathbb{N}$ satisfy $R \geq r^{1+2\epsilon}$. A configuration ζ in B_R is said to satisfy $\zeta \in \text{Fine}$ if the following conditions hold:

- $0 \longleftrightarrow R$;
- the circuit Γ_r exists and satisfies $\Gamma_r \subseteq B_{r^{1+\epsilon}}$;
- the pivotal set $\text{Piv}_{0 \leftrightarrow \Gamma_r} = \text{Piv}_{0 \leftrightarrow R} \cap \text{Int}(\Gamma_r)$ satisfies $|\text{Piv}_{0 \leftrightarrow \Gamma_r}(\zeta)| \leq r^{2(1+2\epsilon)}\alpha_4(r^{1+2\epsilon})$.

Finally, a dynamical configuration $\{\omega_t : t \in \mathbb{R}\}$ is said to satisfy **Fine** if $\omega_0 \in \text{Fine}$.

Definition 4.9. Let $r \in \mathbb{N}$ be even. Let Γ denote a circuit such that $B_r \subseteq \text{Int}(\Gamma)$. Let $a \in \{0, \dots, r/2\}$. The (r, Γ, a) -slim configuration $\chi_{r, \Gamma, a}$ is a particular percolation configuration in $\text{Int}(\Gamma)$, as shown in Figure 4.1, whose set of open hexagons in $\text{Int}(\Gamma) \cap B_{r/2}$ consists of the hexagons in $B_{r/2}$ that intersect the x -axis, and for which $|\text{Piv}_{0 \leftrightarrow \Gamma}| = a$.

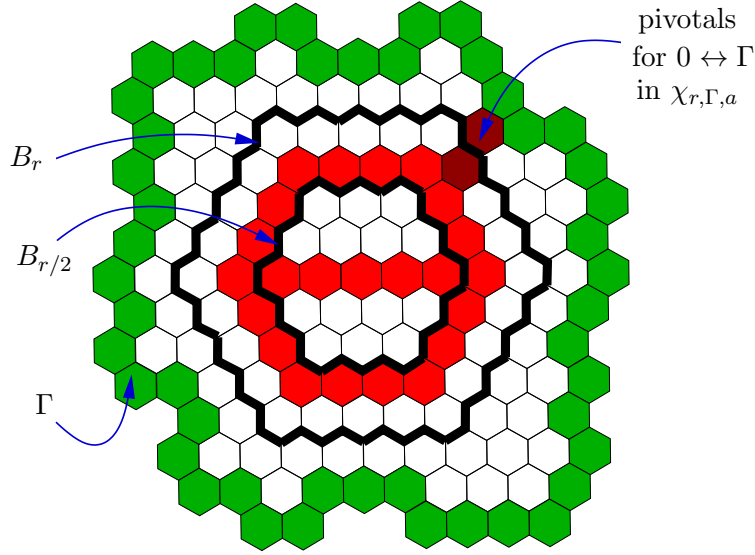


Figure 4.1: The boundary paths delimiting B_2 and B_4 are black, and the circuit Γ is green. The red and dark red hexagons are the open hexagons of $\chi_{4, \Gamma, 2}$: the red hexagonal circuit is set in such a way that its distance from Γ is $a = 2$, and the dark red path is chosen in some arbitrary but fixed way so that it realizes this distance a . Note that this dark red path is the set of pivots for $0 \leftrightarrow \Gamma$.

Definition 4.10. The thinning procedure $\text{Thinning} = \text{Thinning}_r^R$ maps the set of configurations in B_R to itself. Let ζ be such a configuration. If $\zeta \notin \text{Fine}$, then set $\text{Thinning}(\zeta) = \zeta$.

If $\zeta \in \text{Fine}$, let $\text{Thinning}(\zeta)$ be the configuration in B_R of the following form:

$$\text{Thinning}(\zeta)(x) = \begin{cases} \zeta(x) & \text{if } x \in B_R \setminus \text{Int}(\Gamma_r), \\ \chi_{r, \Gamma_r, |\text{Piv}_{0 \leftrightarrow R}|}(x) & \text{if } x \in \text{Int}(\Gamma_r). \end{cases}$$

We define a coupling of \mathbb{P}_{norm} with another dynamical process begun by pairing the initial condition with its thinned counterpart. We denote by ω' the process under \mathbb{P}_{norm} , and write ω'' for the process under the measure \mathbb{P}_{thin} which we now introduce by coupling with \mathbb{P}_{norm} . We set \mathbb{P}_{thin} by choosing its initial condition $\omega''_0 = \text{Thinning}(\omega'_0)$; if the hexagon S selected for initial closure in the definition of \mathbb{P}_{norm} lies in the unbounded component of the complement of $\Gamma_r(\omega'_0)$, we set $S'' = S$; otherwise, we choose S'' uniformly among $\text{Piv}_{0 \leftrightarrow R}(\omega''_0) \cap \text{Int}(\Gamma_r)$. We define ω''_{0+} by modifying ω''_0 by closing S'' . The subsequent evolution of ω'' is made in accordance with the càglàd dynamical updates used in defining ω' . Note that there might be updates that do not have an effect on ω' (the new status coinciding with the old one), and hence are not visible if we see only ω' , while do have an effect on ω'' ; thus ω'' is not entirely measurable w.r.t. ω' , even though the extra randomness in ω'' is quite simple.

We denote by \mathbb{P}_{norm} and \mathbb{P}_{thin} the above dynamics and its thinned counterpart, and write N and T for the reconnection time $\inf \{t > 0 : 0 \xleftrightarrow{\omega_t} R\}$ under \mathbb{P}_{norm} and \mathbb{P}_{thin} . We will often use the above coupling of the two càglàd processes, but will not need a separate notation to denote it. The principal result we need is now stated.

Proposition 4.11 (Thinned versus Normal). *As in Definition 4.8, fix $\epsilon > 0$ small, and consider all large enough $r \in \mathbb{N}$. Then, uniformly in $R \geq r^{1+2\epsilon}$, we have $\frac{\mathbb{E}_{\text{thin}}(T \mathbb{1}_{\text{Fine}})}{\mathbb{E}_{\text{norm}}(N \mathbb{1}_{\text{Fine}})} \rightarrow \infty$ as $r \rightarrow \infty$.*

Proof of Theorem 1.3, assuming Proposition 4.11. We want to show that there exists a circuit Γ in the annulus $A_{r, r^{1+\epsilon}}$ and two configurations ζ' and ζ'' on $\Gamma \cup \text{Int}(\Gamma)$ with $\Gamma = \Gamma_r(\zeta') = \Gamma_r(\zeta'')$, such that $\text{IIC}_R(\zeta') = \text{IIC}_R(\zeta'')$ for each integer $R \geq r^{1+2\epsilon}$, with the common value having a positive limit as $R \rightarrow \infty$, while $\inf_{R \geq r^{1+2\epsilon}} \frac{\text{FETIC}_R(\zeta'')}{\text{FETIC}_R(\zeta')} > 1$.

By Proposition 4.11, we may choose $r \in \mathbb{N}$ so that $\mathbb{E}_{\text{thin}}(T \mathbb{1}_{\text{Fine}}) > 2 \mathbb{E}_{\text{norm}}(N \mathbb{1}_{\text{Fine}})$ for all R sufficiently high. Hence, there exists a choice of circuit Γ in $A_{r, r^{1+\epsilon}}$, and a configuration ζ' in $\text{Int}(\Gamma) \cup \Gamma$, such that $\Gamma = \Gamma_r(\zeta')$, the second and third conditions for Fine occur, and, setting ζ'' equal to the restriction of $\text{Thinning}(\zeta')$ to $\text{Int}(\Gamma) \cup \Gamma$,

$$\mathbb{E}_{\text{norm}}\left(N \mid \omega_0|_{\text{Int}(\Gamma) \cup \Gamma} = \zeta''\right) > 2 \mathbb{E}_{\text{norm}}\left(N \mid \omega_0|_{\text{Int}(\Gamma) \cup \Gamma} = \zeta'\right).$$

It is clear that $\text{IIC}_R(\zeta'') = \text{IIC}_R(\zeta')$; moreover, $\text{IIC}'_R(\zeta'') = \text{IIC}'_R(\zeta')$, since the number of pivotals for $\{0 \leftrightarrow R\}$ is left intact by Thinning. Hence, by Lemma 4.4, we have $\text{FETIC}_R(\zeta'') > 2 \text{FETIC}_R(\zeta')$. \square

The rest of the section will be devoted to the proof of Proposition 4.11. Let us start by collecting the main ingredients needed for the proof; these ingredients will then be proved in the remaining subsections.

Thinning will make a difference only if there is enough time before reconnection for the configuration in $\text{Int}(\Gamma_r)$ to change significantly. To this end, as we will see, the events $\{N > 1/r\}$ and $\{T > 1/r\}$ will be important to us. How different are these two events? Although the set of open hexagons in $\text{Thinning}(\zeta)$ is not exactly a subset of its counterpart for ζ , we can compare the thinned and normal reconnection times in this regime under a certain event **Good**:

$$\text{Good} \cap \{N > 1/r\} \subseteq \{T > 1/r\}, \quad (4.3)$$

where **Good** is defined as follows (and is applied in the above relation to the configuration before thinning):

Definition 4.12. *Let $R, r \in \mathbb{N}$ satisfy $R \geq r^{1+2\epsilon}$ where $\epsilon > 0$ is specified in Definition 4.8. Let ω be a dynamical configuration in B_R . We say that $\omega \in \text{Good}$ if the following conditions are satisfied:*

- $\omega_0 \in \text{Fine}$, as specified in Definition 4.8;
- for each $t \in [0, r^{-1}]$, the inner and outer boundaries of the annulus $A_{r^{1+\epsilon}, r^{1+2\epsilon}}$ are separated by an ω_t -open circuit;
- for each $t \in [0, r^{-1}]$, $0 \xleftrightarrow{\omega_t} r^{1+2\epsilon}$.

Now, to see (4.3), note that the occurrence of **Good** implies that 0 is connected to some open circuit $\Gamma = \Gamma(t)$ such that $B_{r^{1+\epsilon}} \subseteq \Gamma$ for all $0 \leq t \leq r^{-1}$. Hence, $N > 1/r$ implies that $r^{1+\epsilon} \not\leftrightarrow R$ for all $t \in [0, r^{-1}]$ under \mathbb{P}_{norm} . Since the dynamical percolations under \mathbb{P}_{norm} and \mathbb{P}_{thin} agree at all positive times in $A_{r^{1+\epsilon}, R}$, we have that $r^{1+\epsilon} \not\leftrightarrow R$ for all $t \in [0, r^{-1}]$ also under \mathbb{P}_{thin} . Thus, $T > 1/r$ and we obtain (4.3).

The event **Good** is of course useful only if it is reasonably likely to occur. Proposition 4.26, which is the main result of the upcoming Subsection 4.2, will show that

$$\mathbb{P}_{\text{norm}}(\text{Good} \mid N > 1/r) \geq c_1.$$

This, (4.3) and $\text{Good} \subseteq \text{Fine}$ imply the following “stochastic quasi-domination” between T and N :

$$\mathbb{P}_{\text{thin}}(T > 1/r, \text{Fine}) \geq \mathbb{P}_{\text{norm}}(N > 1/r, \text{Good}) \geq c_1 \mathbb{P}_{\text{norm}}(N > 1/r). \quad (4.4)$$

Although the event $\{N > 1/r\}$ has minute probability when R is large, a large portion of the expectation $\mathbb{E}(N \mathbb{1}_{\text{Fine}})$ is contributed by sample points realizing this event. This can be proved using the size-biasing description of the connection time discussed in Lemma 4.5. Indeed, by some rather general size-biasing arguments, together with the uniform boundedness of the expectation $\mathbb{E}(\hat{N}_R) < \infty$ (due to Lemmas 4.5 and 4.6 above), alongside the fact

that $\mathbb{P}_{\text{norm}}(\text{Fine} \mid N > 1/r) \geq c_1$ (due to $\text{Good} \subseteq \text{Fine}$), it will be proved in Subsection 4.3 that

$$\mathbb{E}_{\text{norm}}(N \mid N > 1/r, \text{Fine}) < C_2 < \infty, \quad (4.5)$$

and that

$$\mathbb{P}_{\text{norm}}(\widehat{N\mathbb{1}_{\text{Fine}}} > 1/r) = \frac{\mathbb{E}_{\text{norm}}(N\mathbb{1}_{N>1/r}\mathbb{1}_{\text{Fine}})}{\mathbb{E}_{\text{norm}}(N\mathbb{1}_{\text{Fine}})} > c_2 > 0. \quad (4.6)$$

Finally, as we will prove in Proposition 4.31 of Subsection 4.4, should the dynamics begun under **Thinning** result in at least a short reconnection time, $T > 1/r$, then there is a uniformly positive probability that connection will not be reestablished until very much later:

$$\mathbb{P}_{\text{thin}}(T > g(r) \mid T > 1/r, \text{Fine}) > c_3 > 0, \quad (4.7)$$

for some $g(r) \rightarrow \infty$ as $r \rightarrow \infty$.

Proof of Proposition 4.11. From the above assemblage of facts, we find that

$$\begin{aligned} \mathbb{E}_{\text{thin}}(T\mathbb{1}_{\text{Fine}}) &\geq \mathbb{E}_{\text{thin}}(T\mathbb{1}_{T>1/r}\mathbb{1}_{\text{Fine}}) \\ &= \mathbb{E}_{\text{thin}}(T \mid T > 1/r, \text{Fine}) \mathbb{P}_{\text{thin}}(T > 1/r, \text{Fine}) \\ &\geq c_3 c_1 g(r) \mathbb{P}_{\text{norm}}(N > 1/r, \text{Fine}), \quad \text{by (4.7) and (4.4)} \\ &\geq c_3 c_1 g(r) \frac{\mathbb{E}_{\text{norm}}(N\mathbb{1}_{N>1/r}\mathbb{1}_{\text{Fine}})}{C_2}, \quad \text{by (4.5)} \\ &\geq c_3 c_1 c_2 g(r) \frac{\mathbb{E}_{\text{norm}}(N\mathbb{1}_{\text{Fine}})}{C_2}, \quad \text{by (4.6).} \end{aligned}$$

Therefore, the ratio $\frac{\mathbb{E}_{\text{thin}}(T\mathbb{1}_{\text{Fine}})}{\mathbb{E}_{\text{norm}}(N\mathbb{1}_{\text{Fine}})}$ tends to infinity as $r \rightarrow \infty$, uniformly in $R \geq r^{1+2\epsilon}$, as required. \square

We will now start proving the above ingredients.

4.2 Understanding the law $\mathbb{P}_{\text{norm}}(\cdot \mid N > 1/r)$

In this section, \mathbb{P} will denote the law of càglàd dynamical percolation with time \mathbb{R} . Recall that \mathcal{E}_R is the set of times such that $0 \leftrightarrow R$, now a union of left-open right-closed intervals.

It is hard to understand the conditioned measure $\mathbb{P}' := \mathbb{P}_{\text{norm}}(\cdot \mid N \geq 1/r)$ directly, because the condition has a tiny probability. We will handle this issue by noticing that, for large enough $s \in \mathbb{Z}^+$, we have $\mathbb{P}(\mathcal{E}_R \cap (s/r, (s+1)/r] = \emptyset \mid 0 \in \mathcal{E}_R) > c > 0$, uniformly in $r > 0$ (see Lemma 4.18), and given the existence of this empty interval, $\gamma := \sup\{\mathcal{E}_R \cap [0, s/r)\}$ is a moment such that the reconnection time from it is at least $1/r$. If s is bounded, then the law of dynamical percolation viewed from such a γ (to be denoted by \mathbb{P}'' , see Lemma 4.15) turns out to be not very different from the law \mathbb{P}' (see Lemma 4.16). Therefore, once we prove that ω_t has certain good properties with high probability for all $t \in [0, (s+1)/r]$ under $\mathbb{P}(\cdot \mid 0 \in \mathcal{E}_R, \mathcal{E}_R \cap (s/r, (s+1)/r) = \emptyset)$, which is already a feasible

task, and hence that the dynamical configuration viewed from γ (i.e., the measure \mathbb{P}'') is well behaved, we will be able to deduce almost the same for the measure \mathbb{P}' ; this will be Proposition 4.26, the main goal of this subsection.

Definition 4.13. Call an element $x \in \mathcal{E}_R$ a **marker** if $(x, x + r^{-1}] \cap \mathcal{E}_R = \emptyset$. Write $\mathcal{M} \subseteq \mathcal{E}_R$ for the set of markers. For $x \in \mathcal{M}$, set $\ell_x \geq r^{-1}$ so that $x + \ell_x$ is the first limit point of \mathcal{E}_R encountered to the right of x . Let $s \in \mathbb{Z}^+$ be a (large) integer to be determined later. For each $x \in \mathcal{M}$, set $L_x = [x - sr^{-1}, x - sr^{-1} + \ell_x - r^{-1}]$ if $r^{-1} \leq \ell_x < (s+1)r^{-1}$; if $\ell_x \geq (s+1)r^{-1}$, take $L_x = [x - sr^{-1}, x]$. Define the **domain of attraction** \mathcal{D}_x of $x \in \mathcal{M}$ by $\mathcal{D}_x = L_x \cap \mathcal{E}_R$. See Figure 4.2.

Note that Lemma 4.2 has a straightforward analogue for $\mathbb{P}(\cdot \mid 0 \in \mathcal{M})$, and we have $\mathbb{P}' = \mathbb{P}_{\text{norm}}(\cdot \mid N \geq 1/r) = \mathbb{P}(\cdot \mid 0 \in \mathcal{M})$. We now define the measure \mathbb{P}'' on dynamical configurations on B_R that will be our main tool for understanding \mathbb{P}' .

Definition 4.14. Define the law \mathbb{P}'' so that, for any càglàd dynamical percolation configuration ω satisfying $0 \in \mathcal{M}$,

$$\frac{d\mathbb{P}''}{d\mathbb{P}'}(\omega) = Z^{-1}|\mathcal{D}_0|,$$

where $|\cdot|$ is Lebesgue measure, and $Z > 0$ is a normalization chosen to ensure that \mathbb{P}'' is indeed a probability measure.

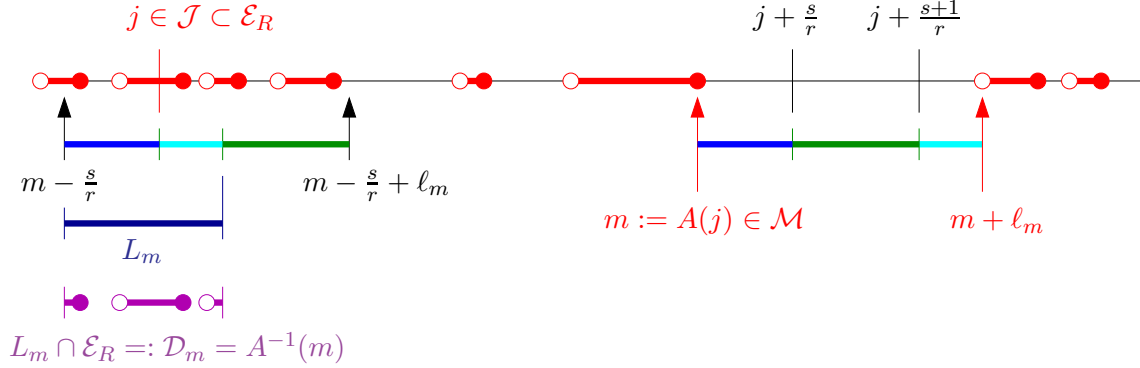


Figure 4.2: The domain of attraction \mathcal{D}_m appearing in the definition of \mathbb{P}'' , and the map $A : \mathcal{J} \rightarrow \mathcal{M}$ appearing in the proof of $\mathbb{P}'' = \tilde{\mathbb{P}}$ (Lemma 4.15).

Lemma 4.15. Let $\tilde{\mathbb{P}}$ denote the following dynamical process. Consider càglàd dynamical percolation $\{\omega_t : t \in \mathbb{R}\}$ in B_R with ω_0 distributed as IIC_R , and with the update decisions made independently of ω_0 . Condition this process on the event that $\mathcal{E}_R \cap (sr^{-1}, (s+1)r^{-1}) = \emptyset$. Let $\gamma \in [0, sr^{-1}]$ be given by $\gamma = \sup\{\mathcal{E}_R \cap [0, sr^{-1}]\}$. Now set $\tilde{\mathbb{P}}$ equal to the conditional law of $\omega(\gamma + \cdot)$. Then $\tilde{\mathbb{P}} = \mathbb{P}''$.

Proof. Under dynamical percolation on B_R , let \mathcal{J} denote the set of times $j \in \mathcal{E}_R$ such that $(j + sr^{-1}, j + (s+1)r^{-1}) \cap \mathcal{E}_R = \emptyset$. Consider the map $A : \mathcal{J} \rightarrow \mathcal{M}$ such that, for each $j \in \mathcal{J}$, $A(j)$ is the largest element of \mathcal{M} preceding $j + sr^{-1}$. Note that $j \in \mathcal{E}_R$ implies that $j \leq A(j) \leq j + sr^{-1}$. Note further that, for each $m \in \mathcal{M}$, we have $A^{-1}(m) = \mathcal{D}_m$. See Figure 4.2.

Consider now an experiment in which, for $x > 0$, dynamical percolation is sampled conditionally on $\mathcal{J} \cap [0, x] \neq \emptyset$, and an element $\chi \in \mathcal{J} \cap [0, x]$ is chosen with the conditional law of normalized Lebesgue measure on this set. Note that, by $\lim_{x \rightarrow \infty} \mathbb{P}(\mathcal{J} \cap [0, x] \neq \emptyset) = 1$, the law of $\omega_{A(\chi)+}$ (using the randomness in both ω and χ has the limit $\tilde{\mathbb{P}}$ as $x \rightarrow \infty$). However, from the previous paragraph we also know that $\omega_{A(\chi)+}$ has a weak limit whose Radon-Nikodym derivative with respect to dynamical percolation given $0 \in \mathcal{M}$ is $|\mathcal{D}_0|$ up to normalization. \square

Lemma 4.16 (Typical events of \mathbb{P}'' will appear in \mathbb{P}'). *The Radon-Nikodym derivative $\frac{d\mathbb{P}''}{d\mathbb{P}'}$ has a second moment that is bounded above by some $B < \infty$ which might depend on the parameter s but not on R . Consequently, $\mathbb{P}'(\mathcal{A}) \geq \mathbb{P}''(\mathcal{A})^2/B$ for any event \mathcal{A} .*

Proof. The claim regarding the Radon-Nikodym derivative follows directly from Lemma 4.17 below. The second claim then follows by Cauchy-Schwarz:

$$\mathbb{P}''(\mathcal{A}) = \int \mathbb{1}_{\mathcal{A}} d\mathbb{P}'' = \int \mathbb{1}_{\mathcal{A}} \frac{d\mathbb{P}''}{d\mathbb{P}'} d\mathbb{P}' \leq \sqrt{\int \mathbb{1}_{\mathcal{A}}^2 d\mathbb{P}'} \sqrt{\int \left(\frac{d\mathbb{P}''}{d\mathbb{P}'}\right)^2 d\mathbb{P}'} \leq \sqrt{\mathbb{P}'(\mathcal{A})} \sqrt{B},$$

as desired. \square

Lemma 4.17. *Let $m_{R,r}$ denote the conditional mean under dynamical percolation of $|\mathcal{E}_R \cap (0, r^{-1})|$ given that this intersection is non-empty. Consider dynamical percolation \mathbb{P} on B_R conditionally on $0 \in \mathcal{M}$. Then the Lebesgue measure of the domain of attraction of the origin satisfies*

$$\mathbb{P}(|\mathcal{D}_0| \geq c m_{R,r} \mid 0 \in \mathcal{M}) \geq cs^{-2} \quad (4.8)$$

and

$$\mathbb{E}(|\mathcal{D}_0|^2 \mid 0 \in \mathcal{M}) \leq Cs^2 m_{R,r}^2, \quad (4.9)$$

for constants $C > c > 0$ which do not depend on r, R or s .

Before starting the proof of Lemma 4.17, we need to verify a basic decorrelation result. In light of Lemma 4.15 (describing \mathbb{P}'' as $\tilde{\mathbb{P}}$), it is far from surprising that this result will be crucial in understanding the measures \mathbb{P}'' and \mathbb{P}' .

Lemma 4.18 (Ensuring an empty interval). *There exists a large $s \in \mathbb{Z}^+$ and a small $c > 0$ such that, for each $r \in \mathbb{Z}^+$ and $R > R_0(r)$, the probability that dynamical percolation with initial condition ω_0 distributed according to $\mathbb{H}C_R$ satisfies $\mathcal{E}_R \cap (sr^{-1}, (s+1)r^{-1}) = \emptyset$ exceeds c .*

An important element of the proof of Lemma 4.18 is the following claim. It is slightly more convenient to reverse time once again, just for this claim. Recall that $\rho(r) = \inf\{s : s^2\alpha_4(s) \geq r\}$, and keep in mind that its magnitude is known to be $r^{4/3+o(1)}$ for percolation on the faces of \mathcal{H} and to lie between $C^{-1}r^{1+\eta}$ and $C r^{1/\eta}$ for some $\eta \in (0, 1)$ and $0 < C < \infty$ for bond percolation on \mathbb{Z}^2 .

Lemma 4.19. *There exists $c > 0$ such that the following holds, independently of $r \in \mathbb{N}$. Let \mathcal{N} denote the event that at no time in the interval $[-r^{-1}, 0]$ is there an open crossing of the annulus $A_{\rho(r), 2\rho(r)}$. For $s > 0$, let \mathcal{Y}_s denote the event that an open crossing of $A_{\rho(r), 2\rho(r)}$ exists at time sr^{-1} . Then, for all large enough $s > 0$ (without dependence on r), we have $\mathbb{P}(\mathcal{N} \cap \mathcal{Y}_s) \geq c$.*

Proof. By considering a coupling in which dynamical updates lead always to the closure of hexagons, we know that $\mathbb{P}(\mathcal{N}) \geq c$ by (1.7), Kesten's result on the near-critical window. Let \mathcal{N}_0 denote the time-0 static event that the conditional probability of \mathcal{N} given the time 0 configuration is at least c . We have that $\mathbb{P}(\mathcal{N}_0) \geq c$ by adjusting the value of $c > 0$. Note then that, denoting by f and g the ± 1 -indicator functions of \mathcal{N}_0 and \mathcal{Y}_0 , and by \hat{f} and \hat{g} their Fourier series, the basic relation (1.11) yields

$$\mathbb{P}(\mathcal{N}_0 \cap \mathcal{Y}_s) - \mathbb{P}(\mathcal{N}_0)\mathbb{P}(\mathcal{Y}_s) = \sum_{S \neq \emptyset} \hat{f}(S)\hat{g}(S) \exp\{-sr^{-1}|S|\}.$$

We apply Cauchy-Schwarz to bound above the absolute value of the right-hand side. Then, the basic relation (1.11) and the decorrelation estimate (1.14) applied to g give the following bound on the resulting term:

$$\begin{aligned} & \left(\sum_{S \neq \emptyset} \hat{f}^2(S) \right)^{1/2} \left(\sum_{S \neq \emptyset} \hat{g}^2(S) \exp\{-2sr^{-1}|S|\} \right)^{1/2} \\ & \leq \epsilon_1 + \left(\sum_{|S| \geq \epsilon_2 r} \hat{g}^2(S) \exp\{-2\epsilon_2 s\} \right)^{1/2} \leq \epsilon_1 + \exp\{-\epsilon_2 s\}, \end{aligned}$$

where ϵ_1 depends on the choice of cutoff $\epsilon_2 > 0$ and may be chosen so that $\epsilon_1 \rightarrow 0$ as $\epsilon_2 \rightarrow 0$. Noting that $\mathbb{P}(\mathcal{N}_0)\mathbb{P}(\mathcal{Y}_s) \geq c_1 > 0$, we see that $\mathbb{P}(\mathcal{N}_0 \cap \mathcal{Y}_s) \geq c_1/2$ by making a suitable choice of ϵ_1 , ϵ_2 and s . Note that $\mathbb{P}(\mathcal{N} \cap \mathcal{Y}_s) \geq c\mathbb{P}(\mathcal{N}_0 \cap \mathcal{Y}_s)$ because \mathcal{N} and \mathcal{Y}_s are conditionally independent given the time-0 configuration. This completes the proof. \square

The next lemma relates the restriction of IIC to a dyadic annulus to the percolation configuration in the annulus obtained by conditioning on an open crossing between the annulus' boundaries.

Lemma 4.20 (Localizing the IIC conditioning). *Let \mathbb{P}_r^R denote the law of critical percolation in $A_{r,R}$ given that $r \longleftrightarrow R$, for $0 \leq r < R \leq \infty$ (where the conditional law*

$\mathbb{P}(\cdot \mid r \leftrightarrow \infty)$ on B_r^c is obtained as a weak limit of $\mathbb{P}(\cdot \mid r \leftrightarrow R)$ as $R \rightarrow \infty$, constructed by [Kes86]). Then, for each $\epsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{A} \in \sigma\{A_{R,2R}\}$ (i.e., an event measurable in the annulus), then $\mathbb{P}_R^{2R}(\mathcal{A}) \geq \epsilon$ implies that $\mathbb{P}_a^b(\mathcal{A}) \geq \delta$, for all $0 \leq a \leq R/2$ and $4R \leq b \leq \infty$ (in particular, for $\text{IIC} = \mathbb{P}_0^\infty$).

Proof. For ζ a configuration in $A_{R,2R}$ such that $R \leftrightarrow 2R$, let $W_{a,R,b}(\zeta)$ denote the conditional probability that $a \leftrightarrow b$ given the occurrence of the events $\omega|_{A_{R,2R}} = \zeta$, $a \leftrightarrow R$ and $2R \leftrightarrow b$. We will argue that for each $\epsilon > 0$ there exists $\delta > 0$ such that, for all large enough $R \in \mathbb{N}$ and $a, b \in \mathbb{N}$ with $0 \leq a \leq R/2$ and $2R \leq b \leq \infty$,

$$\mathbb{P}\left(W_{a,R,b} \leq \delta \mid R \longleftrightarrow 2R\right) \leq \epsilon. \quad (4.10)$$

This easily implies the lemma, as follows. Note that

$$\frac{d\mathbb{P}_a^b}{d\mathbb{P}_R^{2R}}(\zeta) = Z_{a,R,b}^{-1} W_{a,R,b}(\zeta),$$

where $Z_{a,R,b} = \mathbb{P}(a \leftrightarrow b \mid a \leftrightarrow R, R \leftrightarrow 2R, 2R \leftrightarrow b) \leq 1$. Given $\epsilon > 0$, choose by means of (4.10) an $\epsilon' > 0$ such that $\mathbb{P}_R^{2R}(W_{a,R,b} \leq \epsilon') \leq \epsilon/2$ for each $R \in \mathbb{N}$. Thus, if $\mathcal{A} \in \sigma\{A_{R,2R}\}$ satisfies $\mathbb{P}_R^{2R}(\mathcal{A}) \geq \epsilon$, then

$$\mathbb{P}_a^b(\mathcal{A}) = Z_{a,R,b}^{-1} \int_{\mathcal{A}} W_{a,R,b}(\omega) d\mathbb{P}_R^{2R}(\omega) \geq \epsilon' \epsilon / 2,$$

where the inequality follows from restricting the integral to that part of \mathcal{A} on which $W_{a,R,b} > \epsilon'$. Hence the lemma holds with the choice $\delta = \epsilon' \epsilon / 2$.

To prove (4.10), we introduce the function $W_R^\epsilon(\zeta)$ on configurations ζ in $A_{R,2R}$, for $R \in \mathbb{N}$ and $\epsilon \in (0, 1/2)$, which is the conditional probability of $R(1 - \epsilon) \longleftrightarrow 2R(1 + \epsilon)$ under critical percolation given that $\omega|_{A_{R,2R}} = \zeta$.

Lemma 4.21. *For each $\epsilon \in (0, 1/2)$, there exists a constant $c = c_\epsilon > 0$ such that, for each $R, a, b \in \mathbb{N}$ as before and for all configurations ζ in $A_{R,2R}$, we have $W_{a,R,b}(\zeta) \geq c W_R^\epsilon(\zeta)$.*

Proof. Let p_1 denote the probability under critical percolation that there exists an open surrounding circuit in the annulus $A_{R(1-\epsilon),R}$, and let p_2 denote the corresponding probability for the annulus $A_{2R,2R(1+\epsilon)}$. Note that $p_1, p_2 \geq c_\epsilon > 0$ for all R by a simple application of RSW. We claim that

$$W_{a,R,b}(\zeta) \geq p_1 p_2 W_R^\epsilon(\zeta). \quad (4.11)$$

Indeed, consider the conditioning appearing in the definition of $W_{a,R,b}(\zeta)$: under the conditional law, the configuration in $A_{R,2R}^c$ stochastically dominates critical percolation, and thus open surrounding circuits appear in the annuli $A_{R(1-\epsilon),R}$ and $A_{2R,2R(1+\epsilon)}$ with probability at least $p_1 p_2$; the presence of such circuits being an increasing event, the conditional law further conditioned on the presence of such circuits has probability at least $W_R^\epsilon(\zeta)$ of

realizing $R(1 - \epsilon) \longleftrightarrow 2R(1 + \epsilon)$. However, the event $R(1 - \epsilon) \longleftrightarrow 2R(1 + \epsilon)$ and the presence of the two surrounding circuits is enough, alongside the conditions met under the conditional law, to ensure that $0 \longleftrightarrow \infty$. In summary, we obtain (4.11); applying $p_1 p_2 \geq c_\epsilon^2$ completes the proof. \square

Lemma 4.22. *For each $\delta > 0$, there exists $\epsilon_0 > 0$ such that, for all large enough $R \in \mathbb{N}$ and all $\epsilon \in (0, \epsilon_0)$,*

$$\mathbb{P}(W_R^\epsilon \leq 1 - \delta \mid R \longleftrightarrow 2R) \leq \delta.$$

Proof. Note that for any $\delta_1 > 0$ there is an $\epsilon_1 > 0$ such that, for all $\epsilon < \epsilon_1$,

$$\begin{aligned} \mathbb{E}(W_R^\epsilon \mid R \longleftrightarrow 2R) &= \mathbb{P}(R(1 - \epsilon) \longleftrightarrow 2R(1 + \epsilon) \mid R \longleftrightarrow 2R) \\ &= 1 - \frac{\mathbb{P}(R \longleftrightarrow 2R, \text{ but } R(1 - \epsilon) \not\longleftrightarrow 2R(1 + \epsilon))}{\mathbb{P}(R \longleftrightarrow 2R)} \\ &\geq 1 - \delta_1, \end{aligned}$$

because the event $\{R \longleftrightarrow 2R, \text{ but } R(1 - \epsilon) \not\longleftrightarrow 2R(1 + \epsilon)\}$ implies that there are three arms from one side of the annulus $A(R(1 - \epsilon), 2R(1 + \epsilon))$, from radius about ϵR to radius about R , and this event has probability of order ϵ , the 3-arm half-plane probability being of order ϵ^2 . See [Wer09, first exercise sheet].

From this bound, applying Markov's inequality to $1 - W_R^\epsilon$, we get that

$$\mathbb{P}(1 - W_R^\epsilon \geq \sqrt{\delta_1} \mid R \longleftrightarrow 2R) \leq \sqrt{\delta_1},$$

which implies the lemma immediately. \square

Now note that (4.10) follows from Lemmas 4.21 and 4.22 immediately. This completes the proof of Lemma 4.20 (localizing the IIC conditioning) for large enough $R \in \mathbb{N}$; on the other hand, for R bounded, the lemma is trivial. \square

Proof of Lemma 4.18. (Ensuring an empty interval.) Let $\mathcal{C} \in \sigma\{A_{\rho(r), 2\rho(r)}\}$ denote the static event consisting of configurations ζ satisfying $\rho(r) \leftrightarrow 2\rho(r)$ and such that

$$\mathbb{P}(\rho(r) \leftrightarrow 2\rho(r) \text{ at no time in } [sr^{-1}, (s+1)r^{-1}] \mid \omega_0 = \zeta) \geq c.$$

By considering the process $\omega(sr^{-1} - \cdot)$ in Lemma 4.19, we see that

$$\mathbb{P}(\rho(r) \leftrightarrow 2\rho(r) \text{ at time } 0, \rho(r) \leftrightarrow 2\rho(r) \text{ at no time in } [sr^{-1}, (s+1)r^{-1}]) \geq c;$$

in the notation of the statement of Lemma 4.20, we see that $\mathbb{P}_{\rho(r)}^{2\rho(r)}(\mathcal{C}) \geq c$ by reducing the value of $c > 0$. By Lemma 4.20, we infer that for some $\delta > 0$ and for $R \geq 4\rho(r)$, $\text{IIC}_R(\mathcal{C}) > \delta$, as required for the statement of Lemma 4.18. \square

Proof of Lemma 4.17. We start by a simple corollary of Lemma 4.18 concerning the density of markers.

Definition 4.23. Let $\{I_i = (i/r, (i+1)/r) : i \in \mathbb{N}\}$ enumerate the consecutive intervals of length r^{-1} rightwards from the origin. Call any such interval **active** if it has non-empty intersection with \mathcal{E}_R . For any $i \in \mathbb{N}$, call I_i **promising** if I_i is an active interval with the property that \mathcal{M} intersects $\cup_{i \leq j \leq i+s} I_j$.

Lemma 4.24. There exists $c > 0$, independent of R , such that the conditional probability under dynamical percolation B_R given that I_0 is active that I_0 is promising is at least c .

Proof. Let \mathbb{P}_0 denote dynamical percolation on $(0, 1/r)$ weighted according to the size $|\mathcal{E}_R \cap (0, 1/r)|$; under \mathbb{P}_0 , define τ to be an element of $\mathcal{E}_R \cap (0, 1/r)$ with conditional law given by normalized Lebesgue measure on this set. Under \mathbb{P}_0 , the law of dynamical percolation at times $\tau + t$, $t \geq 0$ is, by Lemma 3.3, dynamical percolation started from ΠC_R . By Lemma 4.18, the conditional probability that $(\tau + sr^{-1}, \tau + (s+1)r^{-1}) \cap \mathcal{E}_R = \emptyset$ exceeds some R -independent constant $c > 0$. Whenever this disjointness condition is satisfied, there exists an element of \mathcal{M} somewhere in the interval between τ and $\tau + sr^{-1}$, and thus in the interval $(0, (s+1)r^{-1})$.

We learn that the \mathbb{P}_0 -probability that I_0 is promising exceeds an R -independent constant $c > 0$. Lemma 4.24 will follow once we establish this assertion for dynamical percolation conditioned on the interval I_i being active, a measure we label \mathbb{P}_1 . To make this reduction, it is enough to argue that $\frac{d\mathbb{P}_0}{d\mathbb{P}_1}$ has a bounded second moment, in light of the proof of Lemma 4.16, with the roles of \mathbb{P}'' and \mathbb{P}' being played by \mathbb{P}_0 and \mathbb{P}_1 . By Lemma 2.6, there exists $C > 0$ such that, for all $R > 0$,

$$\begin{aligned} \int \left(\frac{d\mathbb{P}_0}{d\mathbb{P}_1} \right)^2 d\mathbb{P}_1 &= \mathbb{E} \left(|\mathcal{E}_R \cap (0, r^{-1})|^2 \mid \mathcal{E}_R \cap (0, r^{-1}) \neq \emptyset \right) \\ &\leq C \left(\mathbb{E} \left(|\mathcal{E}_R \cap (0, r^{-1})| \mid \mathcal{E}_R \cap (0, r^{-1}) \neq \emptyset \right) \right)^2. \end{aligned}$$

This completes the proof of Lemma 4.24. \square

We can now prove (4.9). Let $\{m_i : i \in \mathbb{N}^+\}$ enumerate the elements of $\mathcal{M} \cap (0, \infty)$ in increasing order. By ergodicity, we have almost surely that

$$\mathbb{E}(|\mathcal{D}_0|^2 \mid 0 \in \mathcal{M}) = \lim_n n^{-1} \sum_{i=2}^n |\mathcal{D}_{m_i}|^2, \quad (4.12)$$

where the term with index $i = 1$ has been harmlessly omitted for later notational convenience. Let $\{\lambda_i\}$ (or $\{\alpha_i\}$) enumerate the indices $i \in \mathbb{N}^+$ of promising (or active) intervals I_i in increasing order. For $i \geq 1$, consider the consecutive intervals I_j beginning the interval after that containing m_i and stopping at the one containing m_{i+1} . Among these, there are at most $s+1$ promising intervals, and $\mathcal{D}_{m_{i+1}}$ is contained in the union

of these promising intervals. Therefore, $\sum_{i=2}^n |\mathcal{D}_{m_i}|^2 \leq (s+1) \sum_{i=2}^{\lambda_{(s+1)^n}} |I_i \cap \mathcal{E}_R|^2$. By Lemma 4.24 and the ergodicity Lemma 2.2, $\lambda_n \leq 2c^{-1}\alpha_n$ for all large enough n . Hence, $\sum_{i=2}^n |\mathcal{D}_{m_i}|^2 \leq (s+1) \sum_{i=1}^{2c^{-1}\alpha_{(s+1)^n}} |I_i \cap \mathcal{E}_R|^2$. By ergodicity again, this upper bound behaves like

$$2c^{-1}(s+1)^2 n \mathbb{E}(|\mathcal{E}_R \cap (0, 1/r)|^2 \mid \mathcal{E}_R \cap (0, 1/r) \neq \emptyset) (1 + o(1))$$

as $n \rightarrow \infty$. Applying Lemma 2.6 to $\mu_R(0, 1/r) = |\mathcal{E}_R \cap (0, 1/r)|$, and using (4.12), we obtain (4.9).

To prove (4.8), in light of (4.9), the Paley-Zygmund second moment method says that it suffices to verify that, for some $c > 0$ and all $R, r, s \in \mathbb{N}_+$,

$$\mathbb{E}(|\mathcal{D}_0| \mid 0 \in \mathcal{M}) \geq c m_{R,r}. \quad (4.13)$$

We now verify this inequality. Let $\rho = \lim_n n^{-1} |\mathcal{M} \cap (0, n)|$ denote the mean number of markers in $[0, 1]$, or, alternatively, $\rho = \mathbb{E}(|\mathcal{M} \cap (0, 1)|)$.

We claim the following.

Lemma 4.25. *Recall that \mathcal{J} denotes the set of times j such that the event $0 \leftrightarrow R$ occurs at time j and at no time in the interval $(j + sr^{-1}, j + (s+1)r^{-1})$. Then $\rho \mathbb{E}(|\mathcal{D}_0| \mid 0 \in \mathcal{M}) = \mathbb{E}(|\mathcal{J} \cap [0, 1]|)$.*

Proof. Recall that the subset \mathcal{J} of \mathcal{E}_R is partitioned into disjoint classes given by domains of attraction \mathcal{D}_m and thus indexed by the set of markers $m \in \mathcal{M}$.

The quantity $\rho \mathbb{E}(|\mathcal{D}_0| \mid 0 \in \mathcal{M})$ is thus the mean Lebesgue measure of the union of the domains of attractions indexed by markers lying in a given unit interval. By the above partition and ergodicity, we arrive at the statement of Lemma 4.25. \square

By translation invariance, $\mathbb{E}(|\mathcal{J} \cap [0, 1]|) = \lim_n n^{-1} \mathbb{E}(|\mathcal{J} \cap [0, n]|)$; by Lemmas 3.2 and 4.18, there exists $c > 0$ such that, for n sufficiently high,

$$n^{-1} \mathbb{E}(|\mathcal{J} \cap [0, n]|) \geq cn^{-1} \mathbb{E}(|\mathcal{E}_R \cap [0, n]|).$$

By translation invariance again, $n^{-1} \mathbb{E}(|\mathcal{E}_R \cap [0, n]|) = \mathbb{E}(|\mathcal{E}_R \cap [0, 1]|)$ which may be written $rm_{R,r} \mathbb{P}(I_0 \text{ is active})$. To summarise the derivation of (4.13) thus far, the preceding inequality and Lemma 4.25 yield

$$\rho \mathbb{E}(|\mathcal{D}_0| \mid 0 \in \mathcal{M}) \geq cr m_{R,r} \mathbb{P}(I_0 \text{ is active}). \quad (4.14)$$

We will show that

$$\rho \leq r \mathbb{P}(I_0 \text{ is active}); \quad (4.15)$$

note then that (4.14) and (4.15) yield (4.13).

To verify (4.15), recall that a marker is by definition an element of \mathcal{E}_R bordered on the right by an interval of length r^{-1} having no intersection with \mathcal{E}_R . Thus, each marker lies

in an active interval, and no active interval contains more than one marker. This implies that the mean rate ρ of markers is at most the mean number of active intervals in a given unit interval, a quantity which may be expressed as $r\mathbb{P}(I_0 \text{ is active})$. This verifies (4.15). This completes the derivation of (4.13) and thus of (4.8), which concludes the proofs of Lemmas 4.17 and 4.16 on the Radon-Nikodym derivative $\frac{d\mathbb{P}''}{d\mathbb{P}}$. \square

We are now ready to address the main goal of this subsection. Recall the notion of **Good** from Definition 4.12.

Proposition 4.26 (\mathbb{P}' is well behaved). *There exists $c > 0$ such that, for any $r > r_0$ and $R > R_0(r)$,*

$$\mathbb{P}(\omega \in \text{Good} \mid 0 \in \mathcal{M}) \geq c.$$

In the proof, we will use the following notion and claim.

Definition 4.27. Fix $\epsilon > 0$ and $r \in \mathbb{N}$ as in Definition 4.8, and let $R \in \mathbb{N}$ satisfy $R \geq r^{1+2\epsilon}$. We say that a dynamical configuration ω in B_R is $\omega \in \text{VeryGood}$ if the following conditions are satisfied:

- $0 \xleftrightarrow{\omega_0} R$;
- For each $t \in [0, (s+1)r^{-1}]$, the inner and outer boundaries of the annulus $A_{r^{1+\epsilon}, r^{1+2\epsilon}}$ are separated by an ω_t -open circuit;
- For each $t \in [0, (s+1)r^{-1}]$, the circuit Γ_r exists and satisfies $\Gamma_r \subseteq B_{r^{1+\epsilon}}$ in ω_t ;
- $|\text{Piv}_{0 \leftrightarrow \Gamma_r}| \leq |\text{Piv}_{0 \leftrightarrow r^{1+\epsilon}}(\omega_t)| \leq r^{2(1+2\epsilon)}\alpha_4(r^{1+2\epsilon})$ for all such t .
- For each $t \in [0, (s+1)r^{-1}]$, $0 \xleftrightarrow{\omega_t} r^{1+2\epsilon}$.

Lemma 4.28. For any $\delta > 0$, there exists $r_0 \in \mathbb{N}$ such that, for all $r \geq r_0$ and $R \geq r^{1+2\epsilon}$,

$$\mathbb{P}(\omega \in \text{VeryGood} \mid 0 \xleftrightarrow{\omega_0} R) \geq 1 - \delta. \quad (4.16)$$

Proof. Let \mathbb{P}_1 denote dynamical percolation in $B_{r^{1+2\epsilon}}$ with ω_0 having the distribution $\mathbb{P}(\cdot \mid 0 \leftrightarrow R)$, and with conditionally independent updates at rate one. Let \mathbb{P}_2 denote the asymmetric dynamical process in $B_{r^{1+2\epsilon}}$ with the same initial distribution as in \mathbb{P}_1 , but with the updates always leading to the closure of hexagons. As usual, we form the obvious coupling \mathbf{Q} of \mathbb{P}_1 and \mathbb{P}_2 such that the first marginal dominates the second for all $t \geq 0$.

In this new notation, the statement of the lemma is equivalent to: for any $\delta > 0$, there exists $r_0 \in \mathbb{N}$ such that, for all $r \geq r_0$ and $R \geq r^{1+2\epsilon}$,

$$\mathbb{P}_1(\omega \in \text{VeryGood}') \geq 1 - \delta, \quad (4.17)$$

where $\text{VeryGood}'$ is given by the second and later conditions defining VeryGood . We claim that, to show (4.17), it is enough that

$$\mathbb{P}_2(\omega \in \text{VeryGood}') \geq 1 - \delta. \quad (4.18)$$

To see that (4.18) is enough for (4.17), note that, under the coupling \mathbf{Q} , it is clear that if the second (\mathbb{P}_2 -distributed) marginal satisfies the second, third and fifth conditions of Definition 4.27, then so does the first (\mathbb{P}_1 -distributed) marginal, because these conditions are monotone. In regard to the fourth condition, write Piv for $\text{Piv}_{0 \leftrightarrow r^{1+\epsilon}}$. Note that if ω_1 and ω_2 are two configurations in $B_{r^{1+\epsilon}}$ such that $\omega_1 \geq \omega_2$ and $0 \leftrightarrow r^{1+\epsilon}$ under ω_2 , then $\text{Piv}(\omega_1) \subseteq \text{Piv}(\omega_2)$: indeed, were a hexagon h in $B_{r^{1+\epsilon}}$ to satisfy $h \in \text{Piv}(\omega_1) \setminus \text{Piv}(\omega_2)$, then its closure would disable $0 \leftrightarrow r^{1+\epsilon}$ in ω_1 but not in ω_2 , a circumstance which stochastic domination prevents. That is, whenever $0 \xrightarrow{\omega_t} r^{1+\epsilon}$ occurs under \mathbb{P}_2 , we have that $|\text{Piv}(\omega_t^1)| \leq |\text{Piv}(\omega_t^2)|$ (where ω^1 and ω^2 denote the \mathbb{P}_1 and \mathbb{P}_2 marginals), and thus (4.18) implies (4.17) and hence (4.16).

It remains to verify (4.18). We start with a simple lemma.

Lemma 4.29. *Let \mathbb{P}_s^\downarrow denote asymmetric dynamical percolation $\{\omega_t : t \geq 0\}$ with ω_0 having the distribution HC_s , then closing hexagons at rate one. Then, restricted to the ball B_r , the Radon-Nikodym derivative $\frac{d\mathbb{P}_R^\downarrow}{d\mathbb{P}_r^\downarrow}(\omega[0, t]^{B_r})$ is bounded from above uniformly in r , $R \geq r$, $t \geq 0$, and all dynamical configurations $\omega[0, t]^{B_r}$.*

Proof. We claim that

$$\frac{d\mathbb{P}_R^\downarrow}{d\mathbb{P}_r^\downarrow}(\omega[0, t]^{B_r}) \leq \frac{\mathbb{P}(0 \leftrightarrow r)\mathbb{P}(r+1 \leftrightarrow R)}{\mathbb{P}(0 \leftrightarrow R)}, \quad (4.19)$$

with the right hand side understood simply in static critical percolation. From this, the lemma follows by quasi-multiplicativity. For the claim, note that the Radon-Nikodym derivatives with respect to asymmetric dynamical percolation \mathbb{P}^\downarrow started from criticality, restricted to B_r , can be written as

$$\frac{d\mathbb{P}_s^\downarrow}{d\mathbb{P}^\downarrow}(\omega[0, t]^{B_r}) = \frac{\mathbb{P}^\downarrow(0 \leftrightarrow s \text{ in } \omega_0 \mid \omega[0, t]^{B_r})}{\mathbb{P}^\downarrow(0 \leftrightarrow s \text{ in } \omega_0)},$$

for any $s \geq r$; in particular, for $s \in \{r, R\}$. On the other hand,

$$\mathbb{P}^\downarrow(0 \leftrightarrow R \text{ in } \omega_0 \mid \omega[0, t]^{B_r}) \leq \mathbb{P}^\downarrow(0 \leftrightarrow r \text{ in } \omega_0 \mid \omega[0, t]^{B_r}) \mathbb{P}^\downarrow(r+1 \leftrightarrow R \text{ in } \omega_0).$$

Since the distribution of ω_0 under \mathbb{P}^\downarrow is simply critical percolation, from the last two displays follows (4.19). \square

Proof of (4.18). Let \mathbb{P}_3 denote asymmetric dynamical percolation $\mathbb{P}_{r^{1+2\epsilon}}^\downarrow$ in $B_{r^{1+2\epsilon}}$, with the notation of the previous Lemma 4.29. By that lemma, it is enough to verify (4.18) with \mathbb{P}_3 in place of \mathbb{P}_2 . We are going to show that each of the four conditions defining **VeryGood'** happens with probability close to 1 if r is large enough.

Let us first look at the four conditions at time zero. The fifth condition (that $0 \leftrightarrow r^{1+2\epsilon}$) is automatically satisfied under \mathbb{P}_3 . The second and third conditions (open circuits in $A_{r^{1+\epsilon}, r^{1+2\epsilon}}$ and in $A_{r^{1+\epsilon}, r}$) are satisfied with high probability in critical percolation by RSW along several scales, and also under the conditioning $0 \leftrightarrow r^{1+2\epsilon}$ by FKG. The fourth condition (there are not too many pivotals for $0 \leftrightarrow r^{1+\epsilon}$) follows from standard quasi-multiplicativity arguments. Namely, as illustrated on Figure 4.3, we have

$$\mathbb{P}(x \in \text{Piv}_{0 \leftrightarrow r^{1+\epsilon}} \mid 0 \leftrightarrow r^{1+2\epsilon}) \asymp \alpha_4(\text{dist}(x, \partial B_{r^{1+\epsilon}}) \wedge \text{dist}(0, x)) \alpha_3(\text{dist}(x, \partial B_{r^{1+\epsilon}}), r^{1+\epsilon}),$$

which can be summed up over the possible hexagons $x \in B_{r^{1+\epsilon}}$ to get

$$\mathbb{E}(|\text{Piv}_{0 \leftrightarrow r^{1+\epsilon}}| \mid 0 \leftrightarrow r^{1+2\epsilon}) = O(1) r^{2(1+\epsilon)} \alpha_4(r^{1+\epsilon}).$$

By quasi-multiplicativity and (1.4), we have

$$\frac{\alpha_4(r^{1+\epsilon})}{\alpha_4(r^{1+2\epsilon})} < C (r^\epsilon)^{2-\eta} \ll r^{2\epsilon},$$

hence Markov's inequality yields

$$\mathbb{P}(|\text{Piv}_{0 \leftrightarrow r^{1+\epsilon}}| > r^{2(1+2\epsilon)} \alpha_4(r^{1+2\epsilon}) \mid 0 \leftrightarrow r^{1+2\epsilon}) \rightarrow 0,$$

as $r \rightarrow \infty$, as desired.

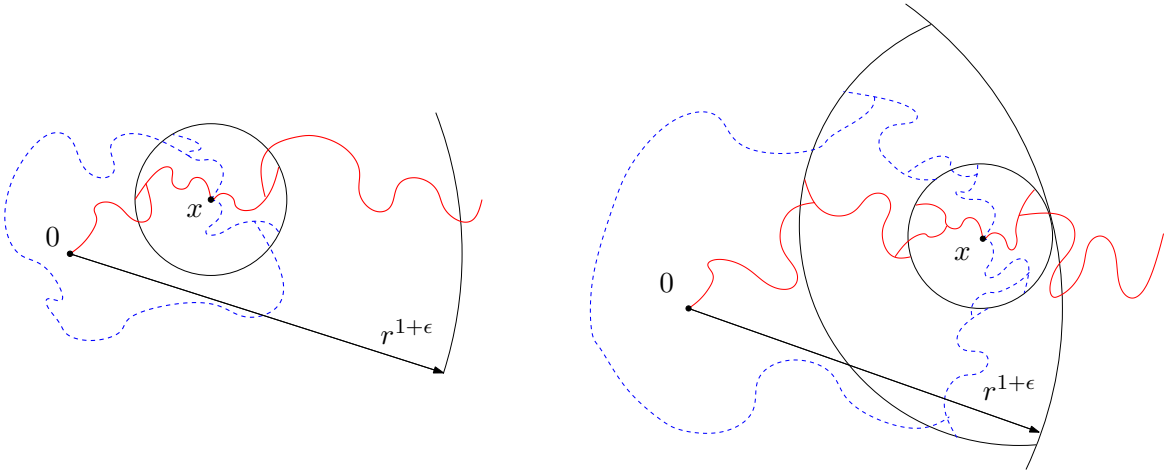


Figure 4.3: Conditioned on $0 \leftrightarrow r^{1+2\epsilon}$, one 4-arm event (first picture) or one 4-arm and one 3-arm events (second picture) are roughly equivalent to x being pivotal for $0 \leftrightarrow r^{1+\epsilon}$.

We now have to prove that the four conditions are also satisfied with high probability at time $t = (s+1)r^{-1}$; then, by the earlier monotonicity argument, we have the result for all $t \in [0, (s+1)r^{-1}]$, as well.

By the exponent bound (1.5) and the choice $(1+2\epsilon)(1-\eta) < 1$ made in Definition 4.8 and onwards, we have that $r^{-1} \ll 1/(r^{2(1+2\epsilon)}\alpha_4(r^{1+2\epsilon}))$, as $r \rightarrow \infty$. Thus, the constant closing of hexagons for time $(s+1)r^{-1}$ keeps the system $B_{r^{1+2\epsilon}}$ well inside the critical window of percolation, established by Kesten, as described in (1.6) and (1.9). Therefore, the above arguments for the second to fourth conditions of Definition 4.27 apply verbatim. The fifth condition can be verified in a similar manner: using (1.9), we have

$$\mathbb{P} \downarrow \left(0 \stackrel{\omega_{(s+1)r^{-1}}}{\longleftrightarrow} r^{1+2\epsilon} \mid 0 \stackrel{\omega_0}{\longleftrightarrow} r^{1+2\epsilon} \right) = 1 - o(1),$$

as $r \rightarrow \infty$. This finishes the proof of (4.18) and Lemma 4.28. \square

Proof of Proposition 4.26. Whenever (4.16) holds, by Lemma 4.18 we also have that

$$\begin{aligned} & \mathbb{P}(\omega \in \text{VeryGood} \mid 0 \stackrel{\omega_0}{\longleftrightarrow} R, \mathcal{E}_R \cap (sr^{-1}, (s+1)r^{-1}) = \emptyset) \\ & \geq \frac{\mathbb{P}(\mathcal{E}_R \cap (sr^{-1}, (s+1)r^{-1}) = \emptyset \mid 0 \in \mathcal{E}_R) - \mathbb{P}(\omega \notin \text{VeryGood} \mid 0 \in \mathcal{E}_R)}{\mathbb{P}(\mathcal{E}_R \cap (sr^{-1}, (s+1)r^{-1}) = \emptyset \mid 0 \in \mathcal{E}_R)} \\ & \geq 1 - \frac{\delta}{c}. \end{aligned} \tag{4.20}$$

Note that if a realization of dynamical percolation in B_R realizes **VeryGood**, then the process $\omega(\gamma + \cdot)$ identified in Lemma 4.15 realizes **Good**. By Lemma 4.15 and (4.20), we find then that $\mathbb{P}''(\omega \in \text{Good}) \geq 1 - \delta/c$. Now Lemma 4.16 implies that an appropriate small choice of δ in (4.16) forces $\mathbb{P}'(\omega \in \text{Good}) \geq c'$ for some absolute constant $c' > 0$, concluding the proof of Proposition 4.26. \square

4.3 Size-biasing arguments

In this subsection, we will prove the bounds (4.5) and (4.6), used in the proof of Proposition 4.11 at the end of Subsection 4.1. To start with, note that

$$\mathbb{P}(\hat{N} > 1/r) = \frac{\mathbb{E}(N \mathbb{1}_{N > 1/r})}{\mathbb{E}N} > c > 0. \tag{4.21}$$

Indeed, by Lemma 4.5, the distribution of $\hat{N} = \hat{N}_R$ stochastically dominates that of FET_R , which implies (4.21) trivially.

Deriving (4.5). Our goal is to show that

$$\mathbb{E}(N \mid N > 1/r) < C < \infty, \tag{4.22}$$

uniformly in r and R for which $R \geq r^{1+2\epsilon}$, since Proposition 4.26 and $\text{Good} \subseteq \text{Fine}$ then imply that, for such values of R and r ,

$$\mathbb{E}(N \mid N > 1/r, \text{Fine}) \leq \frac{\mathbb{E}(N \mid N > 1/r)}{\mathbb{P}(\text{Fine} \mid N > 1/r)} \leq c^{-1} \mathbb{E}(N \mid N > 1/r) < c^{-1}C < \infty,$$

which was the statement of (4.5).

Lemmas 4.5 and 4.6 imply that $\mathbb{E}(\hat{N}) < C$ for some constant $C < \infty$ that is independent of R . This, together with the lower bound (4.21), plugged into the next lemma with $X := N$ and $t := 1/r$, implies (4.22).

Lemma 4.30 (Rough size-biasing). *If X is a non-negative random variable, and $0 < t < 1$ is such that $\mathbb{P}(\hat{X} > t) > c > 0$ and $\mathbb{E}(\hat{X}) < C < \infty$, then $\mathbb{E}(X \mid X > t) < C' < \infty$, where C' depends only on c and C , and not on t .*

Proof. Note that $\mathbb{E}(X \mid X > t) = \mathbb{P}(\hat{X} > t) \frac{\mathbb{E}(X)}{\mathbb{P}(X > t)}$. Hence, we need to show that $\mathbb{E}(X) \leq C' \mathbb{P}(X > t)$. We will need two ingredients for this:

(A) There exists an absolute constant $A < \infty$ such that

$$\mathbb{E}(X \mathbb{1}_{t \geq X}) < A \mathbb{P}(X > t).$$

(B) For all $b > 0$ there is some $K < \infty$ such that

$$\mathbb{E}(X \mathbb{1}_{X > K}) < b \mathbb{E}(X \mathbb{1}_{X > t}),$$

and therefore $\mathbb{E}(X \mathbb{1}_{X > K}) < b' \mathbb{E}(X \mathbb{1}_{K \geq X > t})$ with $b' = b/(1 - b)$.

How would we conclude from here?

$$\begin{aligned} \mathbb{E}(X) &= \mathbb{E}(X \mathbb{1}_{t \geq X}) + \mathbb{E}(X \mathbb{1}_{K \geq X > t}) + \mathbb{E}(X \mathbb{1}_{X > K}) \\ &< A \mathbb{P}(X > t) + K(1 + b') \mathbb{P}(K \geq X > t) \\ &< (A + K(1 + b')) \mathbb{P}(X > t), \end{aligned}$$

and we are done.

Now, for the proof of (A), let us look at

$$\begin{aligned} C \geq \mathbb{E}(\hat{X}) &= \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)} = \frac{\mathbb{E}(X^2 \mathbb{1}_{X > K}) + \mathbb{E}(X^2 \mathbb{1}_{K \geq X > t}) + \mathbb{E}(X^2 \mathbb{1}_{t \geq X})}{\mathbb{E}(X \mathbb{1}_{X > K}) + \mathbb{E}(X \mathbb{1}_{K \geq X > t}) + \mathbb{E}(X \mathbb{1}_{t \geq X})} \\ &\geq \frac{\mathbb{E}(X^2 \mathbb{1}_{X > K})}{\mathbb{E}(X^2 \mathbb{1}_{X > K})/K + K \mathbb{P}(K \geq X > t) + \mathbb{E}(X \mathbb{1}_{t \geq X})}, \end{aligned}$$

hence

$$CK \mathbb{P}(K \geq X > t) + C \mathbb{E}(X \mathbb{1}_{t \geq X}) \geq \left(1 - \frac{C}{K}\right) \mathbb{E}(X^2 \mathbb{1}_{X > K}),$$

for $K > t$ to be fixed later. Assuming the opposite of (A), we have that $\mathbb{E}(X \mathbb{1}_{t \geq X}) \geq A \mathbb{P}(K \geq X > t)$, and the last displayed inequality implies that

$$\left(\frac{CK}{A} + C\right) \mathbb{E}(X \mathbb{1}_{t \geq X}) \geq \left(1 - \frac{C}{K}\right) \mathbb{E}(X^2 \mathbb{1}_{X > K}) \geq \frac{K}{2} \mathbb{E}(X \mathbb{1}_{X > K}),$$

whenever $K \geq 2C$. Therefore,

$$\begin{aligned} c < \frac{\mathbb{E}(X \mathbb{1}_{X > t})}{\mathbb{E}(X)} &\leq \frac{\mathbb{E}(X \mathbb{1}_{K \geq X > t}) + \mathbb{E}(X \mathbb{1}_{X > K})}{\mathbb{E}(X \mathbb{1}_{t \geq X})} \\ &\leq \frac{K \mathbb{P}(K \geq X > t)}{\mathbb{E}(X \mathbb{1}_{t \geq X})} + \frac{2\left(\frac{CK}{A} + C\right)}{K} \leq \frac{K}{A} + \frac{4C}{K}, \end{aligned}$$

whenever $A \geq K$. The first inequality is due to $\mathbb{P}(\hat{X} > t) > c$. By choosing K then A large enough (depending only on c and C), this gives a contradiction, proving (A).

Now, to prove (B), assume that it is not satisfied for some $b > 0$ and an arbitrarily large $K > 0$. Then

$$C \mathbb{E}(X) \geq \mathbb{E}(X^2) \geq \mathbb{E}(X^2 \mathbb{1}_{X > K}) \geq K \mathbb{E}(X \mathbb{1}_{X > K}) \geq bK \mathbb{E}(X \mathbb{1}_{X > t}).$$

For large enough K , this contradicts the bound $\mathbb{P}(\hat{X} > t) > c > 0$, and we are done. \square

Deriving (4.6). Recall that we want to show that $\mathbb{P}(\widehat{N \mathbb{1}_{\text{Fine}}} > 1/r) > c_2 > 0$, uniformly in r and R . Because of the monotonicity in r , it is enough to prove this for some fixed $r = r_0$ (say, $r_0 = 2$). We obviously have

$$\frac{\mathbb{E}(N \mathbb{1}_{N > 1/r_0} \mathbb{1}_{\text{Fine}})}{\mathbb{E}(N \mathbb{1}_{\text{Fine}})} \geq \frac{r_0^{-1} \mathbb{P}(N > 1/r_0, \text{Fine})}{\mathbb{E}N}.$$

We have already noted that Proposition 4.26 implies that $\mathbb{P}(\text{Fine} \mid N > 1/r_0) > c > 0$, hence the numerator is at least $c r_0^{-1} \mathbb{P}(N > 1/r_0)$. For the denominator, in Lemma 4.30 we have proved that $\mathbb{E}N < C' \mathbb{P}(N > 1/r_0)$. Thus we get that the ratio is at least $c r_0^{-1}/C'$, and we are done. \square

4.4 Reconnection from thinned configurations

The missing ingredient in the proof of Proposition 4.11 at the end of Subsection 4.1 is (4.7), namely:

Proposition 4.31 (Things fall apart). *For some $g(r) \rightarrow \infty$ as $r \rightarrow \infty$, we have that*

$$\mathbb{P}_{\text{thin}}(T > g(r) \mid T > 1/r, \text{Fine}) > c_3 > 0.$$

The main step in proving this proposition is:

Proposition 4.32 (The centre cannot hold). *Consider dynamical percolation in B_n with an initial condition in which only the hexagons intersecting the x -axis are open. Then, for some function $g : \mathbb{N} \rightarrow \mathbb{R}^+$ satisfying $g(r) \rightarrow \infty$ as $r \rightarrow \infty$, the probability that at some time between $1/(2n)$ and $g(2n)$ there exists an open path realizing $0 \leftrightarrow n$ is bounded away from one, uniformly in n .*

Proof of Proposition 4.31 assuming Proposition 4.32. Recall the dynamics \mathbb{P}_{thin} specified after Definition 4.10, and note that

$$\begin{aligned} \mathbb{P}_{\text{thin}}(T \leq g(r) \mid T > 1/r, \text{Fine}) &= \mathbb{P}_{\text{thin}}(\exists t \in [1/r, g(r)] : 0 \xleftrightarrow{\omega_t} r \mid T > 1/r, \text{Fine}) \\ &\leq \mathbb{P}_{\text{thin}}(\exists t \in [1/r, g(r)] : 0 \xleftrightarrow{\omega_t} r/2 \mid T > 1/r, \text{Fine}). \end{aligned}$$

Under $\mathbb{P}_{\text{thin}}(\cdot \mid \text{Fine})$, the starting configuration ω_0 is specified in Definition 4.9; inside $B_{r/2}$, this is a deterministic configuration with only the hexagons intersecting the x -axis being open. Since a point mass trivially satisfies the static FKG inequality, we can apply the dynamical FKG inequality Lemma 1.9 for $\mathbb{P}_{\text{thin}}(\cdot \mid \text{Fine})$ inside $B_{r/2}$. Namely, for any $s \in [0, 1]$, consider the dynamical event

$$\mathcal{A}_s := \left\{ [0, \infty) \xrightarrow{\omega} \{0, 1\}^{B_{r/2}} \text{ càglàd} : \mathbb{P}(T > 1/r \mid \omega, \text{Fine}) \geq s \right\}$$

in $B_{r/2}$. This event is decreasing, so that Lemma 1.9 tells us that it is negatively correlated with the increasing event $\{\exists t \in [1/r, g(r)] : 0 \xleftrightarrow{\omega_t} r/2\}$, that is,

$$\begin{aligned} &\mathbb{P}_{\text{thin}}(\mathcal{A}_s \cap \{\exists t \in [1/r, g(r)] : 0 \xleftrightarrow{\omega_t} r/2\} \mid \text{Fine}) \\ &\leq \mathbb{P}_{\text{thin}}(\mathcal{A}_s \mid \text{Fine}) \mathbb{P}_{\text{thin}}(\exists t \in [1/r, g(r)] : 0 \xleftrightarrow{\omega_t} r/2 \mid \text{Fine}). \end{aligned}$$

Integrating over $s \in [0, 1]$ gives

$$\begin{aligned} &\mathbb{P}_{\text{thin}}(\{T > 1/r\} \cap \{\exists t \in [1/r, g(r)] : 0 \xleftrightarrow{\omega_t} r/2\} \mid \text{Fine}) \\ &\leq \mathbb{P}_{\text{thin}}(T > 1/r \mid \text{Fine}) \mathbb{P}_{\text{thin}}(\exists t \in [1/r, g(r)] : 0 \xleftrightarrow{\omega_t} r/2 \mid \text{Fine}). \end{aligned}$$

Summarising,

$$\mathbb{P}(T \leq g(r) \mid T > 1/r, \text{Fine}) \leq \mathbb{P}_{\text{thin}}(\exists t \in [1/r, g(r)] : 0 \xleftrightarrow{\omega_t} r/2 \mid \text{Fine}).$$

By Proposition 4.32, the right hand side is bounded away from one, uniformly in r . This completes the proof. \square

Proof of Proposition 4.32. Let H_n denote the set of hexagons in B_n intersecting the x -axis. The elements of H_n will be labelled $\{h_i : i \in \{-n, \dots, n\}\}$ by the x -coordinate of the triangular lattice point at the centre of the hexagon. We let $B_n \setminus H_n = U_n \cup L_n$ decompose $B_n \setminus H_n$ into its two components above and below the x -axis. The domain U_n

has the shape of a half-hexagon, whose inner boundary naturally decomposes into four paths of hexagons, each along a straight line segment: $H_n \cup \ell_n^1 \cup \ell_n^2 \cup \ell_n^3$, where ℓ_n^2 denotes the horizontal path of hexagons on the top side of U_n (so that the “corner” hexagons containing the points given in complex coordinates by $ne^{i\pi/6}$ and $ne^{i\pi/3}$ belong to ℓ_n^2).

We will denote by \mathbb{P}_{H_n} the dynamical percolation process of the proposition, under which only elements of H_n are open at time 0.

Let \mathcal{C}^{U_n} denote the event that there is a closed path in U_n from ℓ_n^1 to ℓ_n^3 . For each $i \in \{-n/2, \dots, n/2\}$, let $\mathcal{S}_{h_i}^{U_n}$ denote the event that there is a closed path γ in U_n from a hexagon bordering h_i to ℓ_n^2 . The events \mathcal{C}^{U_n} and $\mathcal{S}_{h_i}^{U_n}$ have counterparts \mathcal{C}^{L_n} and $\mathcal{S}_{h_i}^{L_n}$ defined verbatim after reflection in the x -axis. Finally, define

$$\mathcal{T}_+^n := \{\exists i \in \{0, \dots, n/2\} : h_i \text{ is closed}, \mathcal{S}_{h_i}^{U_n}, \mathcal{S}_{h_i}^{L_n}\}$$

and

$$\mathcal{T}_-^n := \{\exists i \in \{-n/2, \dots, 0\} : h_i \text{ is closed}, \mathcal{S}_{h_i}^{U_n}, \mathcal{S}_{h_i}^{L_n}\}.$$

Figure 4.4 illustrates that, for any $t \in (0, \infty)$,

$$\{\omega_t \in \mathcal{C}^{U_n} \cap \mathcal{C}^{L_n} \cap \mathcal{T}_+^n \cap \mathcal{T}_-^n\} \subseteq \{0 \not\leftrightarrow n \text{ in } \omega_t\}. \quad (4.23)$$

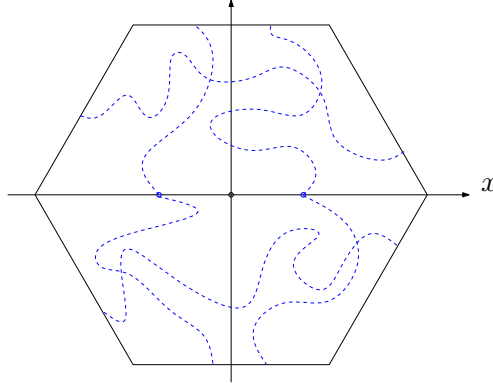


Figure 4.4: The events \mathcal{C}^{U_n} , \mathcal{C}^{L_n} , \mathcal{T}_+^n , \mathcal{T}_-^n .

Given (4.23), Proposition 4.32 will easily follow from the next two lemmas.

Lemma 4.33. *For each $t > 0$,*

$$\mathbb{P}_{H_n} \left(\bigcap_{0 < s < t} \{\omega_s \in \mathcal{C}^{U_n} \cap \mathcal{C}^{L_n}\} \right) \rightarrow 1$$

as $n \rightarrow \infty$.

Proof. Initially the set of open hexagons in U_n is empty; thus, at time s , it has the law of a Bernoulli percolation $\mathbb{P}_{\frac{1}{2}(1-e^{-s})}$. For $0 < s < t < \infty$, let $\omega_{s,t}$ denote the configuration

in which a hexagon is open if the hexagon is open under \mathbb{P}_{H_n} at some time during $[s, t]$. Note then that the marginal law of $\omega_{s,t}$ in U_n is a percolation whose parameter is at most $\frac{1}{2}(1 - e^{-s}) + \frac{1}{2}(1 + e^{-s})(1 - e^{-(t-s)})$. For any given $s > 0$, the percolation parameter of $\omega_{s, s+e^{-s}/2}$ is subcritical. By a standard subcritical percolation estimate, then, for each $s > 0$, $\mathbb{P}_{H_n}(\bigcap_{s < t < s+e^{-s}/2} \{\omega_t \in \mathcal{C}^{U_n}\}) \rightarrow 1$. By a union bound over at most $2se^s$ sets, we see that $\mathbb{P}_{H_n}(\bigcap_{0 < t < s} \{\omega_t \in \mathcal{C}^{U_n}\}) \rightarrow 1$. The statement of the lemma follows by symmetry in the x -axis. \square

Lemma 4.34. *There exists $c > 0$ such that, for all $C > 0$ and for all n sufficiently high,*

$$\mathbb{P}_{H_n} \left(\bigcap_{1/n \leq t \leq C} \{\omega_t \in \mathcal{T}_+^n\} \right) \geq c.$$

Proof. We will argue that, for some $c > 0$, and for all n ,

$$\mathbb{P}_{H_n} \left(\bigcap_{1/n \leq t \leq c} \{\omega_t \in \mathcal{T}_+^n\} \right) \geq c, \quad (4.24)$$

and also that, for any $s \in (0, \infty)$,

$$\lim_{n \rightarrow \infty} \mathbb{P}_{H_n} \left(\bigcap_{s \leq t \leq s+e^{-s}/2} \{\omega_t \in \mathcal{T}_+^n\} \right) = 1. \quad (4.25)$$

Note that (4.24) and (4.25) prove the lemma.

Note that each of the percolations ω_t for $s \leq t \leq s+e^{-s}/2$ is stochastically dominated in $U_n \cup L_n$ by $\omega_{s, s+e^{-s}/2}$ which, as we just noted, is a subcritical percolation, of parameter $p_s < 1/2$.

Let Q_n denote the set of hexagons in \mathcal{H} that lie in the upper-half plane and that intersect the rectangle with vertices $-n^{1/4}e_1$, $n^{1/4}e_1$, $-n^{1/4}e_1 + \frac{n}{2}e_2$ and $n^{1/4}e_1 + \frac{n}{2}e_2$. Let \mathcal{R}_n^+ denote the event that there exists a closed path in Q_n from a hexagon on the top side of Q_n to one that borders h_0 . By [Gri99, Theorem 11.55], for any $p < 1/2$, $\liminf_n \mathbb{P}_p(\mathcal{R}_n^+) > 0$.

Let \mathcal{R}_n^- denote the event \mathcal{R}_n^+ defined after reflection in the y -axis, and let $\mathcal{R}_n = \mathcal{R}_n^+ \cap \mathcal{R}_n^- \cap \{h_0 \text{ is closed}\}$. Clearly, $c = \liminf_n \mathbb{P}_p(\mathcal{R}_n) > 0$. By partitioning $\{0, \dots, n/2\}$ into order $n^{1/4}$ disjoint intervals and considering the analogue of \mathcal{R}_n for each one, we see that

$$\mathbb{P}(\exists i \in \{0, \dots, n/2\} : h_i \text{ is closed, } \mathcal{S}_{h_i}^{U_n} \cap \mathcal{S}_{h_i}^{L_n} \text{ under } \omega_{s, s+e^{-s}/2}) \geq 1 - (1 - c_s)^{n^{1/4}},$$

where for each $s > 0$, $c_s > 0$. Hence, we obtain (4.25).

It is a simple matter to verify (4.24). With a probability that is bounded away from zero uniformly in n , some hexagon h_i , $0 \leq i \leq n/2$, closes during $[0, 1/n]$, and remains closed until at least time one. For some $c > 0$, the marginal of $\omega_{0,c}$ in $U_n \cup L_n$ is a subcritical percolation. Thus, $\mathcal{S}_{h_i}^{U_n} \cap \mathcal{S}_{h_i}^{L_n}$ occurs with positive probability under all ω_s for $0 \leq s \leq c$. This verifies (4.24) and completes the proof of Lemma 4.34. \square

Proof of Proposition 4.32, continued. Note that Lemma 4.34 has a verbatim counterpart for the event \mathcal{T}_-^n . Combining these two lemmas with the aid of the dynamical FKG Lemma 1.9 for the process \mathbb{P}_{H_n} , and using Lemma 4.33, we find that, for any $C > 0$, the left-hand side of (4.23) is satisfied simultaneously for $1/n \leq t \leq C$ with probability tending to one as $n \rightarrow \infty$. Hence, (4.23) proves the result. \square

5 The collapse of the connection near the exceptional set

In this section, we address the question of how quickly the infinite cluster \mathcal{C}_0 in dynamical percolation disintegrates as time varies away from a typical exceptional time. In view of Theorems 1.7 and 1.8, we may rephrase the question as how rapidly this collapse occurs at small positive times in dynamical percolation where ω_0 is chosen to have the law \mathbb{IIC} . In constructing discrete local times in Section 2, we mentioned that there are several natural measurements for how close a finite cluster \mathcal{C}_0 is to being infinite. We write $\text{SIZE}(\mathcal{C}_0)$ as a label for any such notion, and consider three possibilities for it: the volume $|\mathcal{C}_0|$, the radius $\sup\{\|x\| : x \in \mathcal{C}_0\}$, or the “helpfulness” (in providing the event $0 \leftrightarrow \infty$) $\text{HELP}(\mathcal{C}_0) = \tilde{M}_{\mathcal{C}_0}(\omega)$ which was defined in (2.1). Using any of these notions of size, one may try to define a static percolation exponent σ_{SIZE} that measures the robustness of the infinite cluster $\mathcal{C}_0(\omega_0)$, a dynamical percolation exponent δ_{SIZE} that measures how the size of $\mathcal{C}_0(\omega_t)$ degrades with time, and then may try to relate the two exponents, a relation that is expected to reflect the fact that the “speed” of the dynamical process is governed by the number of pivotals in critical percolation. We first give a rough heuristic description of such a general scaling relation; however, since the existence of classical critical exponents is known only for \mathcal{H} , our actual theorem will reformulate the relation in a way that does not use the existence of exponents, and is valid also for the case of \mathbb{Z}^2 .

To understand the robustness of the initial infinite cluster $\mathcal{C}_0(\omega_0)$, we measure the size of its restrictions to finite balls. Thus, we define the **static percolation exponents** by

$$\sigma_{\text{SIZE}} := \lim_{n \rightarrow \infty} \frac{\log \mathbb{E}_{\mathbb{IIC}}(\text{SIZE}(\mathcal{C}_0 \cap B_n(0)))}{\log n}, \quad \text{SIZE} \in \{\text{VOL}, \text{RADIUS}, \text{HELP}\}. \quad (5.1)$$

From [LSW02] and [Kes87b] we know the existence and values of the classical critical exponents

$$\frac{1}{\rho} := \lim_{n \rightarrow \infty} \frac{-\log \mathbb{P}(\text{RADIUS}(\mathcal{C}_0) > n)}{\log n} = \frac{5}{48}, \quad \frac{1}{\delta} := \lim_{n \rightarrow \infty} \frac{-\log \mathbb{P}(|\mathcal{C}_0| > n)}{\log n} = \frac{1}{2\rho - 1} = \frac{5}{91},$$

which imply, with some work, that the exponents (5.1) can be given as

$$\sigma_{\text{VOL}} = \frac{\delta}{\rho} = 2 - \frac{1}{\rho}, \quad \sigma_{\text{RADIUS}} = \frac{\rho}{\rho} = 1, \quad \sigma_{\text{HELP}} = \frac{1}{\rho}.$$

The first one was established in [Kes86, Theorem (8)]. The second one is a triviality. For the third one, an upper bound on $\mathbb{E}_{\mathbb{IIC}}(\text{HELP}(\mathcal{C}_0 \cap B_n))$ follows from (2.4), while a lower bound

can be given by the following argument. Under IIC, the smallest open circuit $\Gamma_{n/2}$ that surrounds $B_{n/2}$ is contained in B_n with a uniform probability $c > 0$. When conditioning on ω^{B_n} , let us restrict ourselves to the part of the probability space where $\Gamma_{n/2} \subset B_n$, condition first on $\omega^{\text{Int}(\Gamma_{n/2})}$, and then, for $R > n$, use the bound

$$\begin{aligned} \mathbb{E}_{\text{IIC}}\left(\mathbb{P}(0 \leftrightarrow R \mid \omega^{B_n})\right) &\geq \mathbb{E}_{\text{IIC}}\left(\mathbb{1}_{\{\Gamma_{n/2} \subset B_n\}} \mathbb{E}_{\text{IIC}}\left(\mathbb{P}(0 \leftrightarrow R \mid \omega^{B_n}) \mid \omega^{\text{Int}(\Gamma_{n/2})}\right)\right) \\ &\geq \mathbb{E}_{\text{IIC}}\left(\mathbb{1}_{\{\Gamma_{n/2} \subset B_n\}} \mathbb{P}(n/2 \leftrightarrow R)\right) \\ &\geq c \mathbb{P}(n/2 \leftrightarrow R) \end{aligned}$$

to find that

$$\mathbb{E}_{\text{IIC}}\left(\lim_{R \rightarrow \infty} \frac{\mathbb{P}(0 \leftrightarrow R \mid \omega^{B_n})}{\mathbb{P}(0 \leftrightarrow R)}\right) \geq \limsup_{R \rightarrow \infty} \frac{c \mathbb{P}(n/2 \leftrightarrow R)}{\mathbb{P}(0 \leftrightarrow R)} \geq c' \mathbb{P}(0 \leftrightarrow n/2)^{-1}.$$

In the first inequality, we used quasi-multiplicativity to obtain the uniform boundedness $\mathbb{P}(0 \leftrightarrow R \mid \omega^{B_n})/\mathbb{P}(0 \leftrightarrow R) \leq C/\mathbb{P}(0 \leftrightarrow n)$ and are thus able to apply the dominated convergence theorem; the second inequality likewise uses quasi-multiplicativity. This concludes the argument for the third equality above.

For the scaling relation, we will also need the static exponent for the number of pivotals for left-right and annulus crossings:

$$\tau := \lim_{n \rightarrow \infty} \frac{\log \mathbb{E}_{p_c} |\text{Piv}_{\mathcal{A}(n)}|}{\log n} = \lim_{n \rightarrow \infty} \frac{\log \mathbb{E}_{p_c} |\text{Piv}_{\mathcal{A}(n, 2n)}|}{\log n} = \frac{3}{4},$$

following from [SmW01], as already mentioned in Subsection 1.4.

Now, we define the **dynamic percolation exponents** by

$$\delta_{\text{SIZE}} = \inf \left\{ y \geq 0 : \liminf_{t \downarrow 0} t^y \text{SIZE}(\mathcal{C}_0(\omega_t)) = 0 \right\},$$

starting the process from ω_0 having the law of IIC. Note that this is a reasonable notion of measuring the collapse of the IIC near ω_0 : time 0 is a limit point of exceptional times, hence $\text{SIZE}(\mathcal{C}_0(\omega_t))$ is infinite along some sequence $t_n \downarrow 0$, but at typical times the cluster is finite and should indeed get smaller with time, according to the following mechanism.

As we will see, for short times $t > 0$, a fragment of the original infinite cluster $\mathcal{C}_0(\omega_0)$ survives at all times $s \in [0, t]$, with the radius of this fragment determined by the maximal scale on which a pivotal hexagon rings during $[0, t]$. As such, we expect that, for any of the above three notions of size,

$$\tau \delta_{\text{SIZE}} = \sigma_{\text{SIZE}}. \quad (5.2)$$

In the interests of concision, we will prove this relation only when $\text{SIZE} = \text{RADIUS}$. The next theorem reformulates the relation in this case, in a way that is valid even for the case of \mathbb{Z}^2 . The rest of the section is devoted to the theorem's proof.

Theorem 5.1. Consider dynamical percolation \mathbb{P}_{IC} with ω_0 having the distribution IC . For $t > 0$, set $\chi_t = \inf_{s \in [0, t]} \text{RADIUS}(\mathcal{C}_0(\omega_s))$. We then have $\log \chi_t \sim \log \rho(1/t)$ \mathbb{P} -a.s. as $t \searrow 0$, with $\rho(\cdot)$ introduced in (1.10). In particular, on \mathcal{H} , we have $\chi_t = t^{-4/3+o(1)}$.

Proof. We start by showing the upper bound on the radius, i.e., by proving that the cluster of the origin falls apart fast enough. The following lemma will be a key step.

Lemma 5.2. There exists $c > 0$ such that the following holds. Let $t \in (0, 1)$ and $r \geq \rho(1/t)$. Let ζ denote a configuration in the annulus $A_{r, 2r}$. Let $N_r^t(\zeta)$ denote the event that the conditional probability of the inner and outer boundaries of $A_{r, 2r}$ not being connected by an open path at time t , given that ω_0 in $A_{r, 2r}$ equals ζ , exceeds $c > 0$. Then $\text{IC}(\{\zeta : N_r^t(\zeta)\}) \geq c$. Moreover, the same conclusion holds for the measure $\text{IC}(\cdot | O \text{ is open})$, where O is any given circuit in $A_{r/4, r/2}$ surrounding $B_{r/4}$, and where $c > 0$ may be chosen independently of O .

Proof. Let \mathcal{C} denote the event that $r \longleftrightarrow 2r$. That

$$\mathbb{P}(\mathcal{C}(\omega_0) \cap \mathcal{C}(\omega_t)^c) \geq c, \quad (5.3)$$

where $c > 0$ is uniform in $r \in \mathbb{N}$ and $t \geq 1/(r^2 \alpha_4(r))$, is a standard and simple consequence of the discrete Fourier analysis approach to critical percolation, already stated as (1.13).

Note that (5.3) implies the statement of the lemma when ω_0 has the law of critical percolation conditioned to have the crossing. To obtain the same statement when ω_0 has the law IC , we can apply Lemma 4.20. For the case of $\text{IC}(\cdot | O \text{ is open})$, we can apply a direct analogue of Lemma 4.20, using $\mathbb{P}_O^\infty = \mathbb{P}(\cdot | O \leftrightarrow \infty)$ in place of \mathbb{P}_a^b . \square

We want to argue that, for any $\epsilon > 0$, we have \mathbb{P}_{IC} -a.s., for all small enough $t > 0$, that

$$\chi_t \leq \rho(1/t) t^{-\epsilon}. \quad (5.4)$$

We define an iterative procedure in an effort to prove (5.4). Let $\ell_1 \in \mathbb{N}$ be minimal such that $2^{\ell_1} \geq \rho(1/t)$. Write $\omega_0^{(\ell_1)}$ for ω_0 restricted to $A_1 := A_{2^{\ell_1}, 2^{\ell_1+1}}$. If $N_{2^{\ell_1}}^t(\omega^{(\ell_1)})$ occurs, and if no open path connects the inner and outer boundaries of A_1 at time t , then the procedure terminates. If one or other of these conditions is unsatisfied, let ℓ_1^* be the minimal $\ell \geq \ell_1 + 1$ such that $A_{2^\ell, 2^{\ell+1}}$ contains an open circuit which encloses B_{2^ℓ} . Set $\ell_2 = \ell_1^* + 2$. Write $A_2 = A_{2^{\ell_2}, 2^{\ell_2+1}}$ and denote by $\omega_0^{(\ell_2)}$ the configuration ω_0 restricted to A_2 . If $N_{2^{\ell_2}}^t(\omega^{(\ell_2)})$ occurs, and if no open path connects the inner and outer boundaries of A_2 at time t , then the procedure terminates. Otherwise, it continues to its next step. The generic step has a similar description to the second one.

Lemma 5.3. Let $J \geq 1$ denote the index of the step at which the procedure terminates. Then there exists $c > 0$ such that, for each $k \in \mathbb{N}$, $\mathbb{P}(\ell_J - \ell_1 \geq k) \leq \exp\{-ck\}$.

Proof. Note that, by Lemma 5.2, there exists $c > 0$ such that $J = 1$ with \mathbb{P}_{IIC} -probability at least c^2 . Under the law \mathbb{P}_{IIC} given the event that either $N_{2^{\ell_1}}^t(\omega^{(\ell_1)})$ does not occur, or $N_{2^{\ell_1}}^t(\omega^{(\ell_1)})$ occurs and $2^{\ell_1} \xleftrightarrow{\omega_t} 2^{\ell_1+1}$, note that the conditional distribution of ω_0 in $B_{2^{\ell_1+1}}^c$ stochastically dominates critical percolation. (This statement is true because it is valid for \mathbb{P}_{IIC} conditionally on an arbitrary configuration in $B_{2^{\ell_1+1}}$ that satisfies $0 \leftrightarrow 2^{\ell_1+1}$ at time zero.) By RSW, FKG and independence on disjoint sets, each dyadic annulus with index at least $\ell_1 + 1$ independently has probability at least $c > 0$ to contain an open circuit disconnecting its boundaries. Thus, conditionally on the value of ℓ_1 , the random variable $\ell_1^* - \ell_1$ is stochastically dominated by a geometric random variable (which we call X_1). Let O_1 denote the innermost of the surrounding open circuits located in $A_{2^{\ell_1}, 2^{\ell_1+1}}$. Conditionally on ω_0 taking a given form on $O \cup \text{Int}(O)$, the conditional distribution of ω_0 in the exterior of O is given by IIC given that O is open. Thus, we may apply the $\text{IIC}(\cdot | O \text{ is open})$ case of Lemma 5.2 to learn that there is probability at least c that $N_{2^{\ell_2}}(\omega^{(\ell_2)})$ occurs. Should this event not occur, or should this event occur alongside the event $2^{\ell_2} \xleftrightarrow{\omega_t} 2^{\ell_2+1}$, then, as previously, the conditional distribution of $\ell_2^* - \ell_2$ is stochastically dominated by a geometric random variable, which we call X_2 .

In this way, we see that $\ell_J - \ell_1$ is stochastically dominated by $\sum_{i=1}^{G_1-1} X_i + 2(G_1 - 1)$, where G_1 is a geometric random variable and $\{X_i : i \in \mathbb{N}\}$ is an independent sequence of i.i.d. geometric random variables. This completes the proof of Lemma 5.3. \square

Note that the inner and outer boundaries of $A_{2^{\ell_J}, 2^{\ell_J+1}}$ are disconnected at time t . Therefore, by Lemma 5.3, $\chi_t \leq \rho(1/t)t^{-\epsilon}$ has probability at least $1 - e^{\log_2(t^{-\epsilon})}$. We have thus verified (5.4).

To complete the proof of Theorem 5.1, it remains to argue that, \mathbb{P}_{IIC} -a.s.,

$$\chi_t \geq \rho(1/t)t^\epsilon \quad (5.5)$$

for all small enough $t > 0$. To prove this, we need the following lemma.

Lemma 5.4. *Let $R \in \mathbb{N}$. For each $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{A} \in \sigma\{B_R\}$ satisfies $\text{IIC}(\mathcal{A}) \geq \epsilon$, then $\text{IIC}_R(\mathcal{A}) \geq \delta$.*

Proof. Recalling the definitions made in (1.1, 2.1), the Bayes' rule computation (2.2), and the quasi-multiplicativity bound (2.4), we have that, for each configuration ζ in B_R realizing $0 \longleftrightarrow R$,

$$\frac{d\text{IIC}}{d\text{IIC}_R}(\zeta) = \frac{\tilde{M}_R(\zeta)}{M_R(\zeta)} \leq C_1,$$

with an absolute constant $C_1 < \infty$. This readily implies the claim. \square

Starting dynamical percolation from IIC_R , and using the coupling in which bits always turn off, Kesten's near-critical one-arm stability (1.9) shows that the probability of still having the connection $0 \longleftrightarrow R$ at all times until $(R^{2(1-\epsilon)}\alpha_4(R^{1-\epsilon}))^{-1}$ is $1 - o(1)$, as

$R \rightarrow \infty$. By Lemma 5.4, the same statement holds when the initial condition is IIC-distributed. From this, (5.5) follows readily. This completes the proof of Theorem 5.1. \square

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